A FOURIER TRANSFORM FOR PARASPHERICAL SPACES OVER FINITE FIELDS

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1. Introduction

Let G be a split reductive group, defined over $\mathbb{F}_q$. Let T be a maximal torus, P be a parabolic subgroup cotnaining T, and M a Levi factor, which we view as both a subgroup of G and a quotient of P. We have an "opposite" parabolic P^{op} such that $P \cap P^{\mathrm{op}} = M$. We let $U = R_u(P)$ be the unipotent radical of P, and $U^{\mathrm{op}} = R_u(P^{\mathrm{op}})$.

It is a theorem of Grosshans [18] that G/U is a strongly quasi-affine variety; i.e., $k[G/U]$ is finitely generated ("strong") and the canonical map $G/U \to \operatorname{Spec} k[G/U]$ is an open embedding ("quasi-affine"). We let $\overline{G/U} := \operatorname{Spec} k[G/U]$, the "affine closure" or "affinization" of G/U. This is generally a singular variety, which we shall call a paraspherical space.[1]

Following the work of Gelfand-Graev[2], Gelfand-Graev-Piatetski-Shapiro [13], Kazhdan [20], and Braverman-Kazhdan [6; 9] on normalized intertwining operators and the mysertious Weyl group action on functions or $\mathcal{D}$-modules on $G/U(B)$ (B a Borel), we hope to establish an involutive Fourier transform

$$\mathcal{F} := \mathcal{F}_{P^{\mathrm{op}},P} : \mathcal{S}(\overline{G/U(P)}(\mathbb{F}_q), \mathbb{C}) \to S(\overline{G/U(P^{\mathrm{op}})}(\mathbb{F}_q), \mathbb{C})$$

where $\mathcal{S}$ means a well-chosen[3] space of $\mathbb{C}$-valued functions on $\mathbb{F}_q$ rational points of $\overline{G/U}$.[4] We would also like to upgrade this, eventually, to a functor between categories of $\overline{\mathbb{Q}}_l$-adic

[1]The homogeneous variety $G/U(P)$ is sometimes called "parabolic basic affine space." However, since this space is neither basic nor affine, we are dissatisfied with this nomenclature. Our preferred terminology is naturally derived from Gelfand's elegant term "horospherical space" for $G/U(B)$ (B a Borel).

[2]It would appear that the first paper on the Weyl-group action on $\mathcal{D}$-modules over G/U is an unpublished article by S. Gelfand and M. Graev from the 1960's (see [16] for some more history). For the action of W on functions on G/U in the adelic setting, see [13], Chapter 3 ("Representations of Adéle Groups), section 5 ("The Space of Horopsheres"), page 361.

[3]At this point, we will be intentionally vague regarding the meaning of "well-chosen." We will be specific in concrete examples.

[4]This particular involution would correspond to the normalized intertwining operator giving the "longest" element of the Weyl Group in the Gelfand-Graev action. Of course, we would, above all, like to construct normalized intertwining operators in general for the spaces $\mathcal{S}(\overline{G/U_i}, \mathbb{C})$, where $U_i = R_u(P_i)$ are the unipotent radicals of the parabolic subgroups P_i containing a fixed Levi factor $M \subset G$. In other words, we would like to find transforms between the function spaces

$$\mathcal{F}_{ji} : \mathcal{S}(\overline{G/U_i}, \mathbb{C}) \to \mathcal{S}(\overline{G/U_j}, \mathbb{C})$$

intertwining the natural $G \times M$-actions, such that

$$\mathcal{F}_{ii} = \mathrm{Id} \text{ and } \mathcal{F}_{kj} \circ \mathcal{F}_{ji} = \mathcal{F}_{ki}.$$

Weil sheaves on $\overline{G/U}$ and $\overline{G/U^{\mathrm{op}}}$, that correspond to the above transformations under Grothendieck's fonctions-faisceaux.

We expect this transform to satisfy a few desiderata:

Ansatz 1). $\mathcal{F}$ must intertwine the natural $G \times M$ action

$$((g,m)[f])\,(x) = f(g^{-1}xm)$$

on the two function spaces.

Observe that this ansatz precludes the transformation from coming from an underlying map of spaces: there is no map $\overline{G/U} \to \overline{G/U^{\mathrm{op}}}$ intertwining the two standard $G\times M$-actions.

Now, there is an natural candidate for a transform satisfying 1); we call this the Radon transform:

$$\mathcal{R} : \mathcal{S}(G/U(P)\,(\mathbb{F}_q), \mathbb{C}) \to \mathcal{S}(G/U(P^{\mathrm{op}})\,(\mathbb{F}_q), \mathbb{C})$$

defined by

$$f \mapsto \mathcal{R}(f) := \left\{ x \mapsto \sum_{\overline{u} \in U^{\mathrm{op}}} f(x\overline{u}) \right\}. \tag{1}$$

This is equivalent to the "pull-push" of the function f along the roof:

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ G/U^{\mathrm{op}} & & G/U. \end{tikzcd}$$

Note that the Radon transform manifestly intertwines the $G\times M$ actions (since M normalizes U). However, the transform is not involutive; that is to say, in general

$$\mathcal{R}'\mathcal{R}(f)(x) = \left\{ x \mapsto \sum_{u\in U} \sum_{\overline{u}\in U^{\mathrm{op}}} f(x\overline{u}u) \right\} \neq f(x).$$

In [9] Braverman-Kazhdan "correct" (i.e., normalize) the Radon transform by convolving the it with a measure on (a subtorus of) $Z(M) \subset T$.[5] Recall that Braverman-Kazhdan are working with functions on spaces of the form $G/[P,P]$.[6] We are considering spaces of the form $G/U(P)$; thus, by analogy, we should be convolving the Radon transform by a measure on the Levi M:

[5]Recall that Braverman-Kazhdan are working over local fields, not finite fields. This is why we use the language of "measures", even though over finite fields, these are the same as "functions."
[6]These have since come to be called "Braverman-Kazhdan" or BK-spaces as in [15; 14].

$$f \mapsto \mathcal{F}(f) := \left\{ x \mapsto \sum_{\overline{u} \in U^{\mathrm{op}}, m \in M} f(xm\overline{u})J(m) \right\}. \tag{2}$$

Notice that if such a transform is to intertwine the M-action, the function J must be central on M; i.e., M-conjugacy-invariant.[7]

However, proceeding exactly as in (2) is unsatisfactory. To see this, we note that even when $G = \mathrm{SL}_2$ – the most basic archetypal case (discussed in greater detail below) – the Fourier transform of Braverman-Kazhdan cannot be written in the form (2). Indeed, recall that the BK Fourier transform [6; 9]

$$\mathcal{F} : \mathcal{S}(\overline{\mathrm{SL}_2/U}(k), \mathbb{C}) \to \mathcal{S}(\overline{\mathrm{SL}_2/U^{\mathrm{op}}}(k), \mathbb{C}),$$

where k is a local field and $\mathcal{S}$ is Braverman-Kazhdan Schwartz space, is just the symplectic Fourier transform on the space of all compactly supported locally constant $\mathbb{C}$-valued functions on $k^2 = \mathbb{A}^2(k)$:

$$\mathcal{F}(f)\left[\begin{pmatrix} b \\ d \end{pmatrix}\right] = \int_{\begin{pmatrix} a \\ c \end{pmatrix} \in \mathbb{A}^2(k)} f\left[\begin{pmatrix} a \\ c \end{pmatrix}\right] \psi(ad - bc)\, da\, dc$$

for all $f \in C_c^\infty(\mathbb{A}^2(k), \mathbb{C})$, where $\psi : \mathbb{F}_q \to \mathbb{C}$ is an additive character. (Recall that $\mathbb{A}^2 \cong \overline{\mathrm{SL}_2/U}$; we will discuss this in more detail below.)

Notice, moreover, that this symplectic Fourier transform for $\mathbb{A}^2$ works perfectly well over finite fields; we replace the integral by a finite sum and the additive Haar measure by the counting measure:

$$\mathcal{F}(f)\left[\begin{pmatrix} b \\ d \end{pmatrix}\right] = \sum_{\begin{pmatrix} a \\ c \end{pmatrix} \in \mathbb{A}^2(\mathbb{F}_q)} f\left[\begin{pmatrix} a \\ c \end{pmatrix}\right] \psi(ad - bc). \tag{3}$$

In particular, this $\mathcal{F}$ is involutive by finite field Fourier inversion. We will take this as the basic model from which we will attempt to generalize to arbitrary G and P. We have:

Ansatz 2). The Fourier transform $\mathcal{F}$ takes the form:

$$\mathcal{F}(f)(y) = \sum_{x \in \overline{G/U}} f(x)\langle x, y \rangle \tag{4}$$

for all $y \in \overline{G/U^{\mathrm{op}}}$, where $\langle -, - \rangle : \overline{G/U}(\mathbb{F}_q) \times \overline{G/U^{\mathrm{op}}}(\mathbb{F}_q) \to \mathbb{C}$ is a $\mathbb{C}$-valued pairing between $\overline{G/U}$ and $\overline{G/U^{\mathrm{op}}}$. The pairing $\langle -, - \rangle$ depends upon choice of an additive character $\psi : \mathbb{F}_q \to \mathbb{C}$. In the case where $G = \mathrm{SL}_2$ and P is the Borel, $\mathcal{F}$ is given by (3).

[7]For a M a torus, as in the case where $P = B$ is a Borel, this property comes for free; tori are commutative.

Now we can see why we ought not to proceed directly as in (2): the (left) action of TU^{op} on SL_2/U is not transitive; however, the pairing $\langle -, - \rangle$ is nonzero everywhere on $\overline{G/U} \times \overline{G/U^{\mathrm{op}}}$ (in particular, it is nonzero on the complement of $yTU^{\mathrm{op}}U$ for any y). Hence we cannot write the symplectic Fourier transform (4) in the form (2).

In the context of [6; 9], this consideration is not significant, since Braverman-Kazhdan are dealing with integration over local fields. As the complement of the "big" TU^{op}-orbit has measure 0, a measure defined on the "big" orbit (via convolving the Haar measure on U^{op} with some measure on T) completely defines a smooth measure on $G/U(k)$. Similarly, Braverman-Kazhdan do not have to contend with the difference between $G/U(k)$ and $\overline{G/U}(k)$, since complement of the former in the latter has measure 0 over local fields.

Over finite fields, however, the rational points of Zariski-closed subsets have positive measure. Thus there are differences between transforms of type (2) as opposed to (4), and between functions on G/U as opposed to functions on $\overline{G/U}$. Appeal to the SL_2 case shows that we should prefer transforms of type (4) and functions on $\overline{G/U}$.

However, we might hope that some remnant of the "normalized" transform of Braverman-Kazhdan can persist in the finite field setting. In particular:

Ansatz 3). If $y \in xUMU^{\mathrm{op}}$, then there should exist a function $M \to \mathbb{C}$ such that, if $y = xum\overline{u}$, then $\langle x, y \rangle = J(m)$. The function J should be invariant under M-conjugacy.

Thus we are insisting that the kernel $\langle -, - \rangle$ restrict to the form (2) when x and y are in the most "generic" relative position.

Finally, we record as a final ansatz the most important property that we expect of our Fourier transform. The whole point of normalization is to obtain transforms between function spaces that compose well (see footnote 4); in our case (dealing with only P and P^{op}), this means involutivity.

Ansatz 4). (Involutivity.[8]) There must be a transform $\mathcal{F}' := \mathcal{F}_{P,P^{\mathrm{op}}} : \mathcal{S}(\overline{G/U(P^{\mathrm{op}})}(\mathbb{F}_q), \mathbb{C}) \to S(\overline{G/U(P)}(\mathbb{F}_q), \mathbb{C})$, defined identically to $\mathcal{F}$ (but swapping the data for P and P^{op}), such that

$$\mathcal{F}' \circ \mathcal{F} = \mathrm{Id}.$$

As noted in footnote 8, if we identify the spaces $\overline{G/U}$ and $\overline{G/U^{\mathrm{op}}}$ (which are isomorphic as varieties, even if no such isomorphism intertwines the standard $G \times M$ actions), the involutive transform $\mathcal{F}$ is then seen to *square* to one. Recall that the standard Fourier transform has *fourth power* one. So perhaps it would be more apropos to call our transform a *Two*rier as opposed to a *Four*ier transform.

The rest of this paper is structured as follows:

In section 2, we discuss the fundamental example of $G = \mathrm{SL}_2$ in detail.

[8] Of course, it is slightly erroneous to speak of $\mathcal{F}$ as involutive, since is is a map between two different spaces. But, as we shall see, these spaces often can be identified with one another, in whichh case, $\mathcal{F}$ will turn out to be ivolutive.

In section 3, we will try to extract certain general ideas from the SL_2 case; this will lead us into discuss the Wang Monoid and the Slipper pairing, which are useful in our efforts to construct a Fourier transform satisfying Ansatz 4).

In section 4, we scrutinize Braverman-Kazhdan's work on normalized intertwining operators in the local field setting. We describe how this motivates the introduction Braverman-Kazhdan's gamma sheaves as the final ingredient in the construction of our Fourier transform. We conclude Section 4 by, finally, giving our proposed construction of the Fourier transform. It is rigged so as to satisfy Ansatz 1), 2) and 3); we state Ansatz 4) (involutivity) as a conjecture. We will also provide a conjectural description of the relevant function space $\mathcal{S}$.

The second half of this paper offers evidence for the involutivity conjecture by analyzing specific cases.

In section 5, we discuss the case or mirabolic subgroups of SL_n. In this case, involutivity follows from finite-field Fourier inversion on a vector space. Here the Schwartz Space consists of all functions on rational points.

In section 6, we discuss the central new result of this paper: a Fourier transform for functions on a certain quadric cone in even-dimensional affine space. The caveat is that we must assume that the function has an average of 0 under the scalar action. We define $\mathcal{S}(X(\mathbb{F}_q), \mathbb{C})$ to be the space of functions subject to this constraint. Then we show that the Fourier transform is indeed involutive. This Fourier transform for quadric cones offers an interesting complement to the recent results of Laumon-Letellier [25], who construct a Fourier transform for functions on the rational points of the *quotient stack* of a quadric cone by the $\mathbb{G}_m$ scalar action, in *odd*-dimensional affine space. Our Fourier transform crucially involves Kloosterman sums in the kernel; involutivity is equivalent to some (possibly new) identities for Kloosterman sums.

In section 7, we see how the result of section 6 implies involutivity of $\mathcal{F}$ for opposite Borels of SL_3 (provided we restrict to functions in $\mathcal{S}$, i.e., whose average under the scalar action is 0). This allows us to construct finite-field normalized intertwining operators for SL_3. This gives us an action of S_3 on $\mathcal{S}$, which appears to be the first example of a family of normalized intertwiners for groups over finite fields (other than the basic case of SL_2).

In section 8, we see how the result of section 6 implies involutivity in the case of Sp_4 and the opposite Siegel parabolics.

2. The Case of SL_2

In this section, let $G = \mathrm{SL}_2$. Then, up to conjugacy, we have only one parabolic, the upper triangular Borel,

$$B := \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subset \mathrm{SL}_2,$$

with unipotent radical $U := U(B) = [B, B]$ given by

$$U := \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subset B.$$

We observe that $G/U \to \mathbb{A}^2$ via the map:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \end{pmatrix},$$

which is well-defined. The map is isomorphic onto its image, which consists of the set of first columns of matrices in SL_2 – that is to say, every nonzero vector. Thus $\mathrm{SL}_2/U \cong \mathbb{A}^2 \setminus \{(0,0)\}$. Moreover, as every regular function on $\mathbb{A}^2 \setminus \{(0,0)\}$ extends canonically to a regular function on $\mathbb{A}^2$ (a computation either verifiable directly or by appeal to the algebraic Hartog Lemma[9]), we see that

$$\overline{\mathrm{SL}_2/U(B)} = \overline{\mathrm{SL}_2/[B,B]} \cong \mathbb{A}^2.$$

Observe that, in particular, $\overline{\mathrm{SL}_2/U}$ is smooth – this is, in fact, quite miraculous; most affinizations of quotients $G/U(P)$ are singular.

Let T denotes the standard torus in SL_2. We have:

$$B^{\mathrm{op}} := \left\{ \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \right\} \subset \mathrm{SL}_2,$$

with unipotent radical $U(B') = [B', B']$ given by

$$U^{\mathrm{op}} := \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\} \subset B.$$

And, as with B, observe that we have a map $G/U^{\mathrm{op}} \to \mathbb{A}^2$ via:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} b \\ d \end{pmatrix},$$

exhibiting $\mathrm{SL}_2/U^{\mathrm{op}}$ as $\mathbb{A}^2 \setminus \{(0,0)\}$, with affine closure $\mathbb{A}^2$.

The critical ingredient in Braverman-Kazhdan's Fourier transform is to notice a G-invariant symplectic duality between these two $\mathbb{A}^2$'s, and then to define the functional transform as a linear (symplectic) Fourier transform, leaving Fourier inversion to automatically take care of involutivity. But observe that there are actually two pairings here, one over $\mathbb{F}_q$ taking values in $\mathbb{A}^1(\mathbb{F}_q)$, and one taking values in $\mathbb{C}$ given by the composition:

$$\overline{\mathrm{SL}_2/U}(\mathbb{F}_q) \times \overline{\mathrm{SL}_2/U^{\mathrm{op}}}(\mathbb{F}_q) \to \mathbb{A}^1(\mathbb{F}_q) \xrightarrow{\psi} \mathbb{C}$$
$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod U, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \bmod U^{\mathrm{op}} \right) \mapsto ad' - cb' \mapsto \psi(ad' - cb')$$

where $\psi : \mathbb{F}_q \to \mathbb{C}$ is an additive character.

[9]Indeed, the case of $\mathbb{A}^2 \setminus \{(0,0)\}$ is perhaps the most well-known example of Hartog's Lemma.

So let us scrutinize the first "$\mathbb{F}_q$" duality between $\overline{\mathrm{SL}_2/U}$ and $\overline{\mathrm{SL}_2/U^{\mathrm{op}}}$, with values in $\mathbb{A}^1$.

Observe that we have an isomorphism between the double-unipotent quotient and $\mathbb{A}^1$:

$$U^{\mathrm{op}}\backslash SL_2/U \cong \mathbb{A}^1, \tag{5}$$
$$\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mapsto a.$$

In particular, we see that the function $a : \mathrm{SL}_2 \to \mathbb{A}^1$ is the only double-unipotent invariant function on SL_2. And observe that $\mathbb{A}^1 = \operatorname{Spec} k[a]$ has a multiplicative monoid structure such that $(\mathbb{A}^1)^\times \cong \mathbb{G}_m \cong T$.

We see that we may define the $\mathbb{A}^1$-pairing via

$$\mathfrak{S} : SL_2/U \times \mathrm{SL}_2/U^{\mathrm{op}} \to U^{\mathrm{op}}\backslash SL_2/U \cong \mathbb{A}^1,$$
$$(gU, hU^{\mathrm{op}}) \mapsto U^{\mathrm{op}}h^{-1}gU$$

which is manifestly G-invariant since $U^{\mathrm{op}}h^{-1}gU = U^{\mathrm{op}}(g'h)^{-1}(g'g)U$.

Indeed:

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad' - cb' & * \\ * & * \end{pmatrix}$$

so that

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bmod U, \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \bmod U^{\mathrm{op}}\right) \mapsto ad' - cb' \tag{6}$$

as desired. Note, furthermore, that if the elements of $G/U(B)$ and $G/U(B')$ come from the same element g of G (i.e., if $a = a', \ldots, d = d'$), then the pairing yields $\det(g) = 1$.

What does this pairing mean? If we are given two elements gU and hU^{op} of G/U and G/U^{op}, respectively, then, generically they will pair to an element of $T = (\mathbb{A}^1)^\times \subset \mathbb{A}^1$. This will be the unique element $t \in T$ such that we have

$$\begin{tikzcd} & g \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ gU & & htU^{\mathrm{op}} \end{tikzcd}$$

under the projections

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ G/U & & G/U^{\mathrm{op}}. \end{tikzcd}$$

Thus saying that the $\mathbb{C}$-valued pairing $\langle -, - \rangle : \overline{G/U} \times \overline{G/\mathcal{U}^{\mathrm{op}}} \to \mathbb{C}$ factors through $\mathfrak{S}$ via

$$\overline{\mathrm{SL}_2/U} \times \overline{\mathrm{SL}_2/U^{\mathrm{op}}} \xrightarrow{\mathfrak{S}} \mathbb{A}^1 \xrightarrow{\psi} \mathbb{C}$$

is precisely the compromise of a transform like (4) with one like (2): we see that, generically, when gU intersects the T-orbit of hU^{op}, the unique t bringing one into contact with the other is given weight $J(t) := \psi(t) = \langle gU, hU^{\mathrm{op}} \rangle$.

Now, observe that we may extend $\mathfrak{S}$ to a map $\overline{\mathrm{SL}_2/U} \times \overline{\mathrm{SL}_2/U^{\mathrm{op}}} \to \mathbb{A}^1$. Identifying the two affine closures with $\mathbb{A}^2$ as above, we see that this pairing becomes, simply, the standard symplectic bilinear form on $\mathbb{A}^2$:

$$\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b' \\ d' \end{pmatrix} \right) \mapsto ad' - cb'.$$

Thus letting $\psi \circ \mathfrak{S} := \langle -, - \rangle$ in (4) gives us the involutive Fourier transform (3) of Braverman-Kazhdan.

3. The Wang Monoid and the Pair of Slipper's

We now are going to generalize what we have found for SL_2. Note that our ultimate goal is to construct a $\mathbb{C}$-valued pairing $\langle -, - \rangle$ such that the transform (4) is involutive. What careful study of the SL_2 example revealed is that, for it to resemble a "normalization" of the Radon transform, it should factor as:

$$\overline{G/U} \times \overline{G/U^{\mathrm{op}}} \xrightarrow{\mathfrak{S}=?} ?? \xrightarrow{???} \mathbb{C}$$

where ?? is another variety over $\mathbb{F}_q$, and ??? is a $\mathbb{C}$-valued function on this space.

In this section, we will offer a potential answer to the question of what the space ?? is and how we might define the map $\mathfrak{S} =?$.

First let us turn our attention to the space ??. In the case of SL_2, we saw that it turned out to be $\mathbb{A}^1$, which contains $\mathbb{G}_m = T$ as an open subset. Thus we expect, in general, that the space ?? should be some kind of enlargement $\overline{M} \supset M$ of the Levi. Even better: we should like $\overline{M}$ to be a reductive monoid whose group of invertible elements is M.

3.1. **The Wang Monoid.**

There is a perfect candidate here, which we will call the Wang monoid. For more details about the Wang monoid, see Wang's article [29]. We will offer just one very quick and elegant definition of this monoid which is particularly well-suited to our purposes. But the Wang monoid is a rich and interesting structure, with several different interpretations, and connections to many areas of mathematics (e.g., Geometric Langlands).

Consider the actions of $U(P)$ and $U(P^{\mathrm{op}})$ on G on the right and on the left respectively. These actions induce actions on the ring of functions, $k[G]$. Then we let

$$\overline{M} := \operatorname{Spec} k[G]^{U(P^{\mathrm{op}}) \times U(P)}.$$

Wang [29] shows that this is an affine, normal, algebraic monoid, with group of units M; he also describes its combinatorial description via its Renner Cone. For other information on the relationship between affine embeddings of $G/U(P)$ and reductive monoids containing M, we refer the reader to [1].

3.2. **The Slipper Pairing.** Now we will turn our attention to the map $\mathfrak{S}$ =?. We will call this map the Slipper Pairing (because of the author's surname, and the natural predilection for Slippers to come in pairs), and let

$$\mathfrak{S} : \overline{G/U(P)} \times \overline{G/U(P^{\mathrm{op}})} \to \overline{M}$$

be given as follows. We first define the restriction:

$$S : G/U(P) \times G/U(P^{\mathrm{op}}) \to \overline{M},$$

$$(gU(P), hU(P^{\mathrm{op}})) \mapsto U(P^{\mathrm{op}})h^{-1}gU(P),$$

and then extend to affine closures. Notice that, generically, $\mathfrak{S}(gU(P), hU(P^{\mathrm{op}})) \in M$, in which case it is the unique element $m \in M$ such that we have

$$\begin{tikzcd} & g \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ gU & & hmU^{\mathrm{op}} \end{tikzcd}$$

under the maps

$$\begin{tikzcd} & G \arrow[dl] \arrow[dr] & \\ G/U(P) & & G/U(P^{\mathrm{op}}). \end{tikzcd}$$

That is to say, it the unique element of $m \in M$ whose right action on hU^{op} makes gU and hmU^{op} "come from" the same element of G. This comports well with the SL_2 case and with Ansatz 4).

4. The Function J

4.1. **Braverman-Kazhdan Normalization.** The last thing we must address before we have fully defined the $\mathbb{C}$-valued pairing $\langle -, - \rangle$ and thus the Fourier transform (4) is the question of the function:

$$J : \overline{M} \to \mathbb{C}.$$

As we have already discussed, for $G \times M$-equivariance to be preserved, this map must be central; i.e., invariant under the action conjugation action of M. (Recall that $M = \overline{M}^{\times}$ so conjugation by M is well-defined.)

All in all, we want a function $J : \overline{M} \to \mathbb{C}$ such that:

1) J is central, and

2) J depends upon an additive character ψ.

Moreover:

3) when $P = B$, a Borel, we expect that J, when restricted to $M = T$, should be analogous to those normalizing factors that are described in Braverman-Kazhdan [9] (since, in the case $P = B$, we have $[P, P] = U(P)$). Moreover, we hope that, in the general case, J (restricted to M), will have at least some kinship with the normalizing factors described in [9] for $G/[B, B]$.

We will therefore closely analyze the normalized intertwining operators of Braverman-Kazhdan as constructed in [9]. In this setting G is a reductive group over a local field k; P and Q are two parabolics subgroups containing the fixed Levi M (such that both have Levi quotients M). We let $\mathcal{O}$ denote the ring of integers of k. Braverman-Kazhdan construct a normalized intertwiner:

$$\mathcal{S}(G/[P,P](k), \mathbb{C}) \to \mathcal{S}(G/[Q,Q](k), \mathbb{C}).$$

Given a graded representation of a dual torus $T^\vee$, Braverman-Kazhdan define a distribution on T, as follows: let $V = \bigoplus_{i \in \mathbb{Z}} V_i$ be a graded finite-dimensional representation of $T^\vee$. We may diagonalize this action, and we choose a homogeneous eigenbasis of $T^\vee$ in V, which we shall write as $\lambda_1, \dots \lambda_k$, where $\lambda_i \in X^*(T^\vee)$, $k = \dim V$, where we permit (somewhat abusively) λ_i and λ_j to equal the same element of $X^*(T^\vee)$ for distinct i and j if that weight has multiplicity in V. Observe that we may consider $\lambda_i \in X_*(T)$ by duality. Let n_i be the homogeneous degree j of λ_i in the grading $V = \bigoplus_{j \in \mathbb{Z}} V_j$. Then we let $\eta_{V,\psi}$ be the distribution on T defined via:

$$\begin{aligned} f \mapsto \int_{\mathbb{G}_m^k} & f(\lambda_1(t_1) \cdots \lambda_k(t_k)) \\ & \cdot \psi(t_1)|t_1|^{n_1/2} \cdots \psi(t_k)|t_k|^{n_k/2} \\ & \cdot \left|\delta_P\left(\lambda_1(t_1) \cdots \lambda_1(t_k)\right)\right|^{1/2} \\ & \cdot d^\times t_1 \cdots d^\times t_k \end{aligned} \tag{7}$$

where $f \in C_c^\infty(T, \mathbb{C})$; $\delta_P(x)$, for $x \in P$, is the module character of P; $d^\times t_i = \frac{q-1}{q} \cdot dt_i/|t_i|$ is the multiplicative Haar measure[10] on $\mathbb{G}_m$ (with q the order of the residue field of $\mathcal{O}$).

Now, we let $M^\vee$ be the Langlands dual group of M (which is defined over $\mathbb{C}$). We have that $(M^{\mathrm{ab}})^\vee = Z(M^\vee)$. Let $\mathfrak{u}_P^\vee$ and $\mathfrak{u}_Q^\vee$ denote the nilpotent radicals of the parabolic subalgebras of $\mathfrak{g}^\vee$ dual to P and Q. Let $\mathfrak{u}_{P,Q}^\vee = \mathfrak{u}_P^\vee / \mathfrak{u}_P^\vee \cap \mathfrak{u}_Q^\vee$. Now, we let e, h and f be a principal $\mathfrak{sl}_2$-triple inside the dual Levi Lie algebra $\mathfrak{m}^\vee$.

Braverman-Kazhdan consider two different graded representations of the torus $Z(M^\vee)$: the representation $\mathfrak{u}_{P,Q}^\vee$ under the adjoint action of $Z(M^\vee)$ and its subrepresentation $(\mathfrak{u}_{P,Q}^\vee)^e$, where V^e for a $\mathfrak{g}^\vee$-rep V means the subspace $\ker(\mathrm{ad}\, e)$. The grading here is given by the

[10] For a group G defined over k, when choosing a (left or right) Haar measure, we will pick the unique such measure that assigns a volume of 1 to $G(\mathcal{O})$.

eigenvalues of $\mathrm{ad}(h)$, the Cartan in our $\mathfrak{sl}_2$-triple. Thus we obtain[11] two different measures on M^{ab}; that of $\eta_{\mathfrak{u}^\vee_{P,Q},\psi}$ and that of $\eta_{(\mathfrak{u}^\vee_{P,Q})^e,\psi}$.

Observe that $[Q,Q]/([Q,Q]\cap[P,P]) = U_Q/(U_P\cap U_Q)$. Let us call this space $U_{Q,P}$. This has a canonical Haar measure du (which we normalize so that $U_{Q,P}(\mathcal{O})$ has volume 1). Now we let $\mathcal{R}: C_c^\infty(G/[P,P](k),\mathbb{C}) \to C^\infty(G/[Q,Q](k),\mathbb{C})$, the Radon transform, be given by:

$$\mathcal{R}(f)(x) = \int_{U_{Q,P}} f(xu)du$$

for all $x \in G/[Q,Q](k)$. We now may convolve this by our measures on M^{ab} to get the two transforms:

$$\mathcal{G}_{Q,P}(f) := \eta_{\mathfrak{u}^\vee_{P,Q},\psi} * \mathcal{R}(f),$$

and

$$\mathcal{F}_{Q,P}(f) := \eta_{(\mathfrak{u}^\vee_{P,Q})^e,\psi} * \mathcal{R}(f).$$

The transform $\mathcal{F}_{Q,P}$ is the Braverman-Kazhdan Fourier transform. They prove that $\mathcal{F}_{Q,P}$, $\mathcal{F}_{P,Q}$ – as well as $\mathcal{G}_{Q,P}$ and $\mathcal{G}_{P,Q}$ – extend to adjoint operators $L^2(G/[P,P],\mathbb{C}) \to L^2(G/[Q,Q])$.

[9] demonstrates another interesting relationship between intertwiners for Borels and the transform $\mathcal{G}$. Observe that in the case of Borels, the $\mathfrak{sl}_2$-triple in the dual torus is trivial, so all the weights n_i in (7) are 0, and $V^e = V$ for any $T^\vee$-rep V. Thus $\mathcal{G}$ and $\mathcal{F}$ agree in this context.

Let $B \subseteq P$ and $B' \subseteq Q$ be two Borels contained in P and Q respectively, such that their relative position w in the Weyl group of G is of minimal length. Then we have canonical projections $\pi: G/[B,B] \to G/[P,P]$ and $\pi': G/[B',B'] \to G/[Q,Q]$. The fibers of these projections are isomorphic to $M/(M\cap P)$, which has an open $U(B)^{\mathrm{op}}\cap M$-orbit. Integrating the action of $U(B)^{\mathrm{op}}\cap M$ with respect to the Haar measure on each fiber, we may define maps $\pi_!: C_c^\infty(G/[B,B],\mathbb{C}) \to C_c^\infty(G/[P,P],\mathbb{C})$ and $\pi'_!: C_c^\infty(G/[B',B'],\mathbb{C}) \to C_c^\infty(G/[Q,Q],\mathbb{C})$. Braverman-Kazhdan show how the following diagram commutes:

$$\begin{tikzcd} L^2(G/[B,B],\mathbb{C}) \arrow[r, "\mathcal{G}_{B',B}"] \arrow[d, "\pi_!"'] & L^2(G/[B',B'],\mathbb{C}) \arrow[d, "\pi'_!"] \\ L^2(G/[P,P],\mathbb{C}) \arrow[r, "\mathcal{G}_{Q,P}"'] & L^2(G/[Q,Q],\mathbb{C}). \end{tikzcd} \tag{8}$$

[11] Notice that a measure η on M^{ab} does indeed act via convolution on functions $f \in C_c(G/[P,P],\mathbb{C})$: say $p: \mathbb{G}_m^k \xrightarrow{\sim} M^{\mathrm{ab}}$ is an isomorphism. We pull back the measure η to $\mathbb{G}_m^k$ along p. Then we may lift p to a homomorphism $\tilde{p}: \mathbb{G}_m^k \to T \subseteq M$, such that composing with the natural projection $M \to M^{\mathrm{ab}}$ yields p. Then we define the action of η via the formula $\int_{\mathbb{G}_m^k} f(g\cdot\tilde{p}(t))[p^*(\eta)](t)$. The integral is independent of the choice of lift $\tilde{p}$ because of f's right-invariance under $[P,P]$.

Observe that, adopting the terminology of [9], if we consider the $Z(M^\vee)$-representation $(\mathfrak{u}^\vee_{P,Q})^{\mathrm{junk}} = \mathfrak{u}^\vee_{P,Q}/(\mathfrak{u}^\vee_{P,Q})^e$, then:

$$\eta_{\mathfrak{u}^\vee_{P,Q},\psi} = \eta_{(\mathfrak{u}^\vee_{P,Q})^{\mathrm{junk}},\psi} * \eta_{(\mathfrak{u}^\vee_{P,Q})^e,\psi} \tag{9}$$

and

$$\mathcal{G}_{Q,P} = \eta_{(\mathfrak{u}^\vee_{P,Q})^{\mathrm{junk}},\psi} * \mathcal{F}_{Q,P}. \tag{10}$$

Now we return to our original problem. Notice, firstly, that in the cases we are considering, $Q = P^{\mathrm{op}}$, whence $\mathfrak{u}^\vee_{P,Q} = \mathfrak{u}^\vee_P$, the Lie algebra of $U(P^\vee)$. For our purposes, we will adopt the philosophy that we we should "keep the junk" – i.e., if one wants to intertwine function spaces (or sheaves) on $G/U(P)$ as opposed to $G/[P,P]$, then we should prefer the operator $\mathcal{G}$ to $\mathcal{F}$ (and correspondingly prefer the representation $\mathfrak{u}^\vee_{P,Q}$ to $(\mathfrak{u}^\vee_{P,Q})^e$ and the distribution $\eta_{\mathfrak{u}^\vee_{P,Q},\psi}$ to $\eta_{(\mathfrak{u}^\vee_{P,Q})^e,\psi}$).

In addition to the fact that the operator $\mathcal{G}$ satisfies the elegant property (8), there is another, more philosophical reason to adopt the perspective of favoring $\mathcal{G}$ to $\mathcal{F}$ for our purposes. This is that, in general, there seems to be a natural affinity

$$G/[P,P] \leftrightsquigarrow (\mathfrak{u}^\vee_P)^e$$
$$G/U(P) \leftrightsquigarrow (\mathfrak{u}^\vee_P).$$

See, for instance, the computation of the IC sheaves of the two corresponding Drinfeld compactifications [5].

Of course, we desire a distribution on M (or, in fact on the Wang monoid $\overline{M}$). The distributions in Braverma-Kazhdan of the form $\eta_{V,\psi}$ (for V a $T^\vee$-representation) only live on the torus M^{ab}. We would like to find the analogue of the distributions $\eta_{\mathfrak{u}^\vee_P,\psi}$ on the whole Levi M – and then to extend it to $\overline{M}$. And, of course, we would like to do this over finite fields.

A perfect candidate suggests itself, from (somewhat independent) work of Braverman-Kazhdan: the γ-sheaves for reductive groups [8].

4.2. **The Definition of the γ-sheaf.** The BK γ-sheaf $\Phi_{G,\rho,\psi}$ is sheaf on G, a reductive group, determined by a representation ρ of the Langlands dual group $G^\vee \to \mathrm{GL}_n(\overline{\mathbb{Q}_l})$ and an additive character ψ. These sheaves are constructed over $\overline{\mathbb{Q}_l}$, so we will pick an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}_l}$ to translate this into $\mathbb{C}$-valued functions.

$\Phi_{G,\rho,\psi}$ is constructed as follows[12]: let $T^\vee \subset G^\vee$ be a maximal torus. Restricting ρ to $T^\vee$ we may diagonalize the representation ρ with respect to the weights of $T^\vee$, which gives us a multi-set of characters $\{\lambda_i\}$, where $\lambda_i : T^\vee \to \mathbb{G}_m$. By duality, these define a multi-set of cocharacters $\lambda_i : \mathbb{G}_m \to T$. We consider the diagram:

[12]Here we are following Cheng-Ngô [12].

$$\begin{tikzcd} & \mathbb{G}_m^n \arrow[dl, "\mathrm{Tr}"'] \arrow[dr, "p_{\underline{\boldsymbol{\lambda}}}"] & \\ \mathbb{A}^1 & & T \end{tikzcd}$$

where Tr is given by summing the coordinates in each factor of $\mathbb{G}_m$, and $p_{\underline{\boldsymbol{\lambda}}}$ is given by the product of the maps $\lambda_i : \mathbb{G}_m \to T$. (Note that there are $n = \dim \rho$ of the λ_i, counting each with multiplicity; thus each factor of $\mathbb{G}_m$ corresponds to a λ_i and we define $p_{\underline{\boldsymbol{\lambda}}} : (x_1, \dots, x_n) \mapsto \lambda_1(x_1) \cdots \lambda_n(x_n) \in T$.) Let $\mathcal{L}_\psi$ be the Artin-Schreier sheaf on $\mathbb{A}^1$ associated to the additive character ψ. We define the sheaf $\Phi_{T,\rho,\phi}$ on T via

$$\Phi_{T,\rho,\psi} := p_{\underline{\boldsymbol{\lambda}}!}\mathrm{Tr}^*(\mathcal{L}_\psi)[n]\left(\frac{n}{2}\right)$$

This sheaf is perverse local system over its support (which is the subtorus of T given by the image of $p_{\underline{\boldsymbol{\lambda}}}$). Moreover, when the λ_i satisfy a technical condition called σ-positivity (which will prevail in the cases of interest to us) it is irreducible and isomorphic to $p_{\underline{\boldsymbol{\lambda}}*}\mathrm{Tr}^*(\mathcal{L}_\psi)[n]\left(\frac{n}{2}\right)$. It is sometimes called the Kloosterman sheaf, the hypergeometric sheaf, or the γ-sheaf for tori.

We now define an action of the Weyl group W of (G, T) on the sheaf $\Phi_{T,\rho,\psi}$. This is the most subtle part of defining the γ-sheaf. Let V_ρ denote the underlying vector $\overline{\mathbb{Q}_\ell}$-space of the representation ρ. Diagonalizing V_ρ with respect to $T^\vee$, we obtain the decomposition:

$$V_\rho = \bigoplus_{i=1}^m V_{\lambda_i}.$$

Let $n_i = \mathrm{dim} V_{\lambda_i}$, the multiplicity of λ_i in V_ρ. Of course $n_1 + \cdots + n_m = n$. Define:

$$\underline{\boldsymbol{\lambda}} = (\lambda_1, \dots, \lambda_1, \dots, \lambda_m, \dots \lambda_m) \in X_*(T)^n$$

with each λ_i appearing n_i times.

Now, let us pick a basis $\{v_{i,j}, 1 \le j \le n_i\}$ for V_{λ_i}, for each $i = 1, \dots m$. Let $A_i = \{v_{i,j}\}$ for $1 \le j \le n_i$, and let $A = A_1 \sqcup \ldots \sqcup A_m$. We see that A is a basis for V_ρ; and we may identify the Weyl group of $GL(V_\rho)$ with the symmetric group $S_n = \mathrm{Perm}(A)$. We define

$$S_{\underline{\boldsymbol{\lambda}}} = S_{n_1} \times \cdots \times S_{n_m} \subseteq S_n$$

to be the subgroup of $\sigma \in \mathrm{Perm}(A)$ such that $\sigma(A_i) = A_i$.

We now consider a slightly larger subgroup of S_n, which we will denote $S'_{\underline{\boldsymbol{\lambda}}}$. This subgroup allows us to permute the subset A_i in addition to the elements within each A_i. In particular it consists of those $\xi \in \mathrm{Perm}(A)$ such that there exists $\sigma \in S_m$ (thought of as acting on the indices $i = 1, \dots m$) such that $\xi(A_i) = A_{\sigma(i)}$. Observe that such a ξ can only exist if $n_i = |A_i| = |A_{\sigma(i)}| = n_{\sigma(i)}$ for all i. We obtain an exact sequence of groups

$$1 \longrightarrow S_{\underline{\boldsymbol{\lambda}}} \longrightarrow S_{\underline{\boldsymbol{\lambda}}'} \longrightarrow S_m,$$

where the image of the last map consists of those permutations of $\{1,\ldots,m\}$ preserving the function $i \mapsto n_i$.

Letting W be the Weyl group of G, we see that there is a map $\rho_W : W \to S'_{\underline{\boldsymbol{\lambda}}}/S_{\underline{\boldsymbol{\lambda}}} \subseteq S_m$. We construct an extension W' of W by $S_{\underline{\boldsymbol{\lambda}}}$ as in the diagram

$$\begin{tikzcd} 1 \arrow[r] & S_{\underline{\boldsymbol{\lambda}}} \arrow[r] \arrow[d] & W' \arrow[r] \arrow[d] & W \arrow[r] \arrow[d, "\rho_W"] & 1 \\ 1 \arrow[r] & S_{\underline{\boldsymbol{\lambda}}} \arrow[r] & S'_{\underline{\boldsymbol{\lambda}}} \arrow[r] & S'_{\underline{\boldsymbol{\lambda}}}/S_{\underline{\boldsymbol{\lambda}}} \arrow[r] & 1 \end{tikzcd}$$

We see that W' is the subgroup of $W \times S'_{\underline{\boldsymbol{\lambda}}}$ consisting of those (w,ξ) such that $\xi \mapsto \rho_W(w)$ under the natural projection $S'_{\underline{\boldsymbol{\lambda}}} \to S'_{\underline{\boldsymbol{\lambda}}}/S_{\underline{\boldsymbol{\lambda}}}$. The map $S_{\underline{\boldsymbol{\lambda}}}$ sends $\sigma \to (1,\sigma)$.

The homomorphism $p_{\underline{\boldsymbol{\lambda}}} : \mathbb{G}_m^n \to T$ is W' equivariant; here W' acts on the the factors of $\mathbb{G}_m^n$ via its image in $S'_{\underline{\boldsymbol{\lambda}}} \subseteq S_n$, and W' acts on T via its projection to W. Thus we have a canonical isomorphism

$$\iota'_{w'} : (w')^*\Phi_{T,\rho,\psi} \xrightarrow{\sim} \Phi_{T,\rho,\psi}$$

for all $w' \in W'$. However, we prefer a twisted version of this isomorphism. For $w' = (w,\xi) \in W'$, we let $\mathrm{sign}(w') = \mathrm{sign}(w)\mathrm{sign}(\xi)$, where $(w) = \pm 1$ is given by the sign of $\rho(w)$ in S_m, and $\mathrm{sign}(\xi)$ is the sign of ξ in S_n (recall that $\xi \in S_{\underline{\boldsymbol{\lambda}}} = S_{n_1} \times \cdots \times S_{n_m} \subseteq S_n$). We now define our preferred action of W' on $\Phi_{T,\rho,\psi}$:

$$\iota_{w'} = \mathrm{sign}(w') \cdot \iota'_{w'} : (w')^*\Phi_{T,\rho,\psi} \xrightarrow{\sim} \Phi_{T,\rho,\psi},$$

where multiplication by $\mathrm{sign}(w')$ means the scalar multiplication of the morphism $\iota'_{w'}$ by ± 1.

We next consider the Grothendieck-Springer simultaneous resolution:

$$\begin{tikzcd} \widetilde{G} \arrow[r, "\widetilde{c}"] \arrow[d, "\widetilde{q}"'] & T \arrow[d, "q"] \\ G \arrow[r, "c"'] & T /\!\!/ W \end{tikzcd}$$

where

$$\widetilde{G} = \{(g, hB) \in G \times G/B \,:\, h^{-1}gh \in B\},$$

c is the Steinberg morphism, $\widetilde{c} : (g,hB) \mapsto \pi(h^{-1}gh)$ (where $\pi : B \twoheadrightarrow T$ is the canonical projeciton), q is the obvious projection, and $\widetilde{q} : (g,hB) \mapsto g$. Notice that if we restrict everything to the regular semisimple locus (of both T and G), the diagram becomes Cartesian: the upper row is a W-torsor over the lower.

Now we consider the sheaf on G given by pull-pushing $\Phi_{T,\rho,\psi}$ along the Grothendieck-Springer fibration:

$$\mathrm{Ind}_T^G(\Phi_{T,\rho,\psi}) := \widetilde{q}_!\,\widetilde{c}^*(\Phi_{T,\rho,\psi})[d],$$

where $d = \dim G - \dim T$. Because the Grothendieck-Springer map $\widetilde{q}$ is small and proper, we see that $\mathrm{Ind}_T^G(\Phi_{T,\rho,\psi})$ is a perverse sheaf on G. Let $j^{\mathrm{rss}} : G^{\mathrm{rss}} \to G$ denote the open inclusion of the regular semisimple locus of G. We see that

$$\mathrm{Ind}_T^G(\Phi_{T,\rho,\psi}) \cong j_{!*}^{\mathrm{rss}} j^{\mathrm{rss}*} \mathrm{Ind}_T^G(\Phi_{T,\rho,\psi}).$$

Since $\Phi_{T,\rho,\psi}$ is W-equivariant (under ι_w), and $\widetilde{G}^{\mathrm{rss}} \to G^{\mathrm{rss}}$ is a W-torsor, W acts on $j^{\mathrm{rss}*}\mathrm{Ind}_T^G(\Phi_{T,\rho,\psi})$. Thus, by functoriality of the intermediate extension, W acts on $\mathrm{Ind}_T^G(\Phi_{T,\rho,\psi})$.

At last we may define the γ-sheaf.

Definition 4.1. *The γ-sheaf of ρ on G, written $\Phi_{G,\rho,\psi}$, is defined to be the W-invariant direct factor of* $\mathrm{Ind}_T^G(\Phi_{T,\rho,\psi})$.

We will denote this simply as Φ_ρ when G and ψ are clear. Observe that

$$\Phi_\rho = j_{!*}^{\mathrm{rss}} j^{\mathrm{rss}*}\left(\Phi_\rho\right),$$

so we may offer another definition of Φ_ρ as follows. If we restrict the sheaf $\Phi_{T,\rho,\psi}$ to T^{rss}, we see that it is a W-equivariant perverse sheaf on T. Thus it descends to a perverse local system on $T^{\mathrm{rss}} /\!\!/ W$, which we shall call Φ_λ. Then

$$\Phi_\rho \cong j_{!*}^{\mathrm{rss}} c^{\mathrm{rss}*}\Phi_\lambda.$$

Thus we see that the γ-sheaf is, in essence, a Kloosterman sheaf on the maximal torus, extended by conjugacy-invariance to the regular semisimple locus G^{rss} of G, and then extended (via the "intermediate extension") to G. Thus "generically" the associated function is a convolution of $\psi(t)$'s, like in the normalizing factors of Braverman-Kazhdan's normalized intertwining operators.

4.3. **Construction of the Fourier Transform.** Let the representation ρ denote the adjoint representation of $M^\vee$ on $\mathfrak{u}_P^\vee$, and consider the corresponding γ-sheaf $\Phi_{M,u_P^\vee,\psi}$. The choice of $\mathfrak{u}_P^\vee$ under the adjoint action is part of our "keep the junk" philosophy described above: we are trying to emulate the distribution $\eta_{\mathfrak{u}_P^\vee,\psi}$ and the transform $\mathcal{G}$.

$\Phi_{M,u_P^\vee,\psi}$ is, of course, merely a sheaf on M; we *really* want a sheaf on the Wang monoid $\overline{M}$. Of course, if we wish to preserve perversity, there is a canonical way to do this. Thus, letting $j : M \to \overline{M}$ be the open inclusion, we define:

$$\mathfrak{J} := j_{!*}(\Phi_{M,u_P^\vee,\psi}). \tag{11}$$

Now, at last, we may state our conjecture.

Definition (Sheaf Version). Let G be a reductive group, with parabolic subgroup P, Levi factor M, and maximal torus T. Let $\overline{M}$ be the Wang Monoid of M, $\mathfrak{S}$ be the Slipper pairing, and $\mathfrak{J}$ the γ-sheaf on $\overline{M}$ defined above. Consider the diagram:

$$\begin{tikzcd} & \overline{G/U(P^{\mathrm{op}})} \times \overline{G/U(P)} \arrow[r, "\mathfrak{S}"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \overline{M} \\ \overline{G/U(P^{\mathrm{op}})} & & \overline{G/U(P)} \end{tikzcd}$$

We define the transform

$$\mathcal{F}_{P^{\mathrm{op}},P} \,:\, D_c^\flat\left(\overline{G/U(P)}, \overline{\mathbb{Q}_l}\right) \to D_c^\flat\left(\overline{G/U(P^{\mathrm{op}})}, \overline{\mathbb{Q}_l}\right)$$

given by

$$\mathcal{F}(\mathscr{S}) = R\pi_{1\,!}\big(\pi_2^*(\mathscr{S}) \otimes \mathfrak{S}^*(\mathfrak{J})\big). \tag{12}$$

We now define a class of "special" sheaves and functions; these will be the sheaves/functions on which we expect the Fourier transform to be involutive. This definition is motivated by numerical evidence from specific examples [13].

Definition 4.2. *We say that a sheaf $\mathscr{S}$ is "special" if $\pi_!(\mathscr{S}) = 0$ for the quotient $\pi : \overline{G/U} \to (\overline{G/U})/\mathbb{G}_m$, the latter viewed as a stack, where the $\mathbb{G}_m$ action is given as follows. Consider the roots $\{\alpha^\vee\}_P$ whose 1-parameter subgroup lies in $U(P)^\vee$. Let $\Pi_P^\vee$ denote those coroots in $\{\alpha^\vee\}_P$ which are not decomposable into a positive linear combination of other coroots in $\{\alpha^\vee\}_P^\vee$. Let $\check{\alpha}_P$ denote the sum of such $\alpha^\vee \in \Pi_P^\vee$. Then $\mathbb{G}_m \xrightarrow{\check{\alpha}_P} Z(M) \subset T \subset M$ acts on $\overline{G/U}$ on the right.*

Similarly, we define a function $f : \overline{G/U}(\mathbb{F}_q) \to \mathbb{C}$ to be "special" if

$$\sum_{t \in \mathbb{F}_q^\times} f(x\alpha_P^\vee(t)) = 0 \tag{13}$$

for all $x \in \overline{G/U(P)}(\mathbb{F}_q)$. We will let $\mathcal{S}$ denote the class of special sheaves/functions.

Observe that the property of specialness – i.e., of being annihilated when summed over the action of a torus – is similar to the defining property of cuspidal automorphic forms, which are annihilated when integrating over the action of a unipotent subgroup. In fact,

[13] See section 6 and 7. It appears that, more generally, for an intertwiner $G/U(P) \to G/U(P')$, we may define the coroot $\alpha_{P',P}^\vee$ given by the sum of all "simple" coroots whose 1-parameter subgroup is contained in $U(P)^\vee \cap U(P'^{\mathrm{op}})\vee$. That is to say, we consider the collection of coroots of G whose 1-parameter subgroup lies in $U(P)^\vee \cap U(P'^{\mathrm{op}})^\vee$. We let $\Pi_{P,P'}^\vee$ denote those coroots in this set that are not the positive linear combination of two such. Then we let $\alpha_{P',P}^\vee$ be the sum of the coroots $\Pi_{P,P'}^\vee$. This in turn gives us a right action of $\mathbb{G}_m$ on $\overline{G/U(P)}$ via $\mathbb{G}_m \xrightarrow{\check{\alpha}_P} Z(M) \subset T \subset M$, which has a natural right action on $\overline{G/U}$. Thus we may define π to be the canonical morphism of stacks $\overline{G/U} \to \overline{G/U(P)}/\mathbb{G}_m$, and we may say that a sheaf $\mathscr{S}$ is special with respect to the $P \rightsquigarrow P'$ intertwiner if $\pi_!(\mathscr{S}) = 0$. We expect that restricting to these sheaves will permit us to compose intertwiners well – we will see this in the case of SL_3.

so-called "toroidal" automorphic forms (those with toral average 0, as in our case) have appeared in, e.g., the work of Zagier [30].

We do not yet have a spectral interpretation of the notion of "specialness" – however, we can observe the following: in the case of $\overline{\mathrm{SL}_2/U}$, a special function is one whose average under the scaling action on $\mathbb{A}^2(\mathbb{F}_q)$ is 0. If we consider the scalar averaging map $\mathrm{Fun}(\mathbb{A}^2(\mathbb{F}_q)) \to \mathrm{Fun}(\mathbb{P}^1(\mathbb{F}_q) \cup (*/\mathbb{G}_m)(\mathbb{F}_q))$, we see that taking this kernel is forcing $f(0) = 0$, and killing off a copy of Steinberg $\oplus$ Trivial from the representation $\mathrm{Fun}(\mathbb{A}^2(\mathbb{F}_q))$.

However, as this example suggests, we are not quite satisfied with having to restrict our spaces of functions and sheaves: we did not have to do so in the above case of SL_2. Indeed involutivity of the 2-dimensional symplectic Fourier trnasform applies to all functions and sheaves on $\mathbb{A}^2$ (as it simply follows from Fourier inversion for the Fourier-Deligne transform). So we would *really* like the Fourier transform to apply to all sheaves/functions, as it does in the case of SL_2. As we will see in Section 5, we do not need to restrict to special functions to achieve involutivity in the case of the mirabolic subgroup of SL_n – here, as in the special case of SL_2 and the Borel, the Fourier transform reduces to the usual linear Fourier transform. It is worth noting that in the case SL_2 and the Borel, and more generally mirabolics for SL_n, the parapsherical space is smooth (in fact a linear affine space). In all the cases we shall encounter below, special functions are only supported on the smooth locus of $\overline{G/U}$. [14] It is possible that our Fourier transform is a "principal term" of some more general Fourier transform which does apply to all sheaves and functions – we might imagine that the boundary terms relate to the algebraic geometry of the singularities of $\overline{G/U}$. Alternatively, we suspect that our Fourier transform may "lift" to a Fourier transform on some desingularization of $\overline{G/U(P)}$, where, we hope, involutivity applies without restriction. [15] In the meantime, we are uncertain if "specialness" – in spite of its intriguing relations to "special" representations like the Steinberg, and Zagier's toroidal automorphic forms – is the "right" condition on our sheaves/functions.

We may now state our main conjecture:

Conjecture (Sheaf Version). The transform $\mathcal{F}$ defines, up to a Tate twist and dimension shift, an involution on special sehaves. More precisely:

$$\mathcal{F}_{P,P^{\mathrm{op}}} \circ \mathcal{F}_{P^{\mathrm{op}},P}(\mathscr{S}) \cong \mathscr{S}[n](d).$$

We have also the "function" version of the above:

Definition (Function Version). Let $J(x)$, for $x \in \overline{M}(\mathbb{F}_q)$, denote the trace of Frobenius on the stalk $\mathfrak{J}_x$, with values taken in $\mathbb{C}$ (via some isomorphism $\overline{\mathbb{Q}_l} \cong \mathbb{C}$ discussed above). I.e., let J denote the function corresponding to the sheaf $\mathfrak{J}$. Then we let:

$$\mathcal{F}_{P^{\mathrm{op}},P} : \mathrm{Fun}(\overline{G/U(P)}(\mathbb{F}_q), \mathbb{C}) \to \mathrm{Fun}(\overline{G/U(P^{\mathrm{op}})}(\mathbb{F}_q), \mathbb{C})$$

given by

[14] However, merely being supported on the smooth locus is not alone sufficient to cause our Fourier transform to be involutive; see the formulae (22) in Section 6 for the double Fourier transformation of a general Dirac mass.

[15] The author is attempting to verify this idea at the time of writing for SL_n and the so-called Laumon resolution [23; 22].

$$\mathcal{F}_{P^{\mathrm{op}},P}(f)(y) = \sum_{x \in \overline{G/U}} f(x) J(\mathfrak{S}(x,y)) \tag{14}$$

This defines $G \times M$-equivariant transform of function spaces.

Conjecture (Function Version). Let f be a special $\mathbb{C}$-valued function on $\overline{G/U(P)}(\mathbb{F}_q)$. Then $\mathcal{F}$ defines an involution (up to multiplication by a power of q) on f; i.e.:

$$\mathcal{F}_{P,P^{\mathrm{op}}} \circ \mathcal{F}_{P^{\mathrm{op}},P}(f) = q^{2d} \cdot f.$$

5. The Mirabolic

Let $G = \mathrm{SL}_n$. The mirabolic ("miraculous parabolic") subgroup of SL_n is given by the $(n-1)+1$ block parabolic:

$$P = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \dots & 0 & a_{n,n} \end{pmatrix} \right\}.$$

The unipotent radical is isomorphic to the vector space k^{n-1}:

$$U(P) = \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 & a_{1,n} \\ 0 & 1 & \cdots & 0 & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_{n-1,n} \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \right\}.$$

The Levi $M \cong \mathrm{GL}_{n-1}$ is given by:

$$P = \left\{ \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & 0 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & 0 \\ 0 & 0 & \dots & 0 & D^{-1} \end{pmatrix} \right\}.$$

where $D = \det[a_{i,j}]_{1 \le i,j \le n-1}$. Since D is determined by the upper left $(n-1) \times (n-1)$-block, we will often write M as just the collection of matrices $[a_{i,j}]_{1 \le i,j \le n-1}$.

We see that the quotient $G/U(P)$ isomorphic to the locus of matrices in $\mathrm{Mat}_{n,n-1}$ of full rank (i.e., rank $= n-1$):

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \dots & a_{n,n-1} & a_{n,n} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & * \\ 0 & 1 & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & * \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$
$$\mapsto \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} \\ a_{n,1} & a_{n,2} & \dots & a_{n,n-1} \end{pmatrix}.$$

Indeed, the image of this GL_n under this map is $\mathrm{Mat}_{n,n-1}^{\mathrm{rk}=n-1} \subset \mathrm{Mat}_{n,n-1}$. On the other hand the quotient variety $G/U(P)$ has dimension $(n^2-1)-(n-1) = n(n-1) = \dim \mathrm{Mat}_{n,n-1}$. Moreover, we can verify that the map is separable and bijective on points, proving the isomorphism of varieties. The degenerate locus in $\mathrm{Mat}_{n,n-1}$ has codimension 2 if $n=2$, and codimension $n-1$ if $n \geq 3$. In all these cases the codimension is ≥ 2, whence, by Hartog's Lemma we see that the affine closure $\overline{G/U(P)} \cong \mathrm{Mat}_{n,n-1}$. Note that this is smooth (and in fact an affine space!). This is highly atypical – part of the the miracle of mirabolic subgroup.

Next we describe the space $\overline{G/U(P^{\mathrm{op}})}$, the Wang Monoid, and the Slipper pairing. $\overline{G/U^{\mathrm{op}}}$ can be naturally identified $\mathrm{Mat}_{n-1,n}$ as follows: consider $(gU^{\mathrm{op}})^{-1} = U^{\mathrm{op}}g^{-1}$. Then we have:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ * & * & \dots & * & 1 \end{pmatrix} \begin{pmatrix} a^{-1}_{1,1} & a^{-1}_{1,2} & \cdots & a^{-1}_{1,n-1} & a^{-1}_{1,n} \\ a^{-1}_{2,1} & a^{-1}_{2,2} & \cdots & a^{-1}_{2,n-1} & a^{-1}_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a^{-1}_{n-1,1} & a^{-1}_{n-1,2} & \dots & a^{-1}_{n-1,n-1} & a^{-1}_{n-1,n} \\ a^{-1}_{n,1} & a^{-1}_{n,2} & \dots & a^{-1}_{n,n-1} & a^{-1}_{n,n} \end{pmatrix}$$
$$\mapsto \begin{pmatrix} a^{-1}_{1,1} & a^{-1}_{1,2} & \cdots & a^{-1}_{1,n-1} & a^{-1}_{1,n} \\ a^{-1}_{2,1} & a^{-1}_{2,2} & \cdots & a^{-1}_{2,n-1} & a^{-1}_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \\ a^{-1}_{n-1,1} & a^{-1}_{n-1,2} & \dots & a^{-1}_{n-1,n-1} & a^{-1}_{n-1,n} \end{pmatrix}.$$

Now, the Wang monoid is by definition $\mathrm{Spec}\, k[G]^{U(P)\times U(P^{\mathrm{op}})}$. We see that the double-unipotent-invariants are a_{ij}, for $1 \leq i,j \leq n-1$:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ * & * & \dots & * & 1 \end{pmatrix} \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \dots & a_{n,n-1} & a_{n,n} \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 & * \\ 0 & 1 & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & * \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n-1} & * \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n-1} & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & * \\ * & * & \dots & * & * \end{pmatrix}$$

Thus the Wang Monoid is $\mathrm{Mat}_{n-1,n-1}$ (and observe that the Levi, isomorphic to GL_{n-1}, is indeed an open subset).

Under these identifications the Slipper pairing $\mathfrak{S}(gU(P), hU(P^{\mathrm{op}})) := U(P^{\mathrm{op}})h^{-1}gU(P)$ is simply given by the multiplication of matrices:

$$\begin{aligned} \mathrm{Mat}_{n,n-1} \times \mathrm{Mat}_{n-1,n} &\to \mathrm{Mat}_{n-1,n-1} \\ (M, N) &\mapsto NM. \end{aligned} \tag{15}$$

Now, we will compute the function J. To do this we must calculate the weight decomposition of the adjoint representation of $M^\vee$ on $\mathfrak{u}_P^\vee$. Recall that the Langlands dual of SL_n is PGL_n, and that the dual of GL_{n-1} is GL_{n-1}. Thus the weights of PGL_n are given by

$$\left\{ \sum_i a_i \pi_i : \sum a_i = 0 \right\}$$

where $\pi_i : \mathrm{diag}(t_1, \dots, t_n) \mapsto t_i$. We see that the representaiton $\mathfrak{u}_P$ has weights $\{\pi_i - \pi_n\}_i$ for $i = 1, \dots n-1$. These correspond to the cocharacters $e_i - e_n : \mathbb{G}_m \to T_M = T_G$, where $e_i : t \to \mathrm{diag}(1, \dots, t, \dots, 1)$, with t in the ith position.

Restricting to the upper $(n-1) \times (n-1)$ block, we see that the coweights in GL_{n-1} are simply e_i, for $i = 1, \dots, n-1$. Thus we have the diagram:

$$\begin{tikzcd} & \mathbb{G}_m^{n-1} \arrow[dl] \arrow[dr] & \\ \mathbb{A}^1 & (t_1, \dots t_{n-1}) \arrow[dl, mapsto] \arrow[dr, mapsto] & T_{\mathrm{GL}_{n-1}} \\ t_1 + \cdots + t_{n-1} & & \mathrm{diag}(t_1, \dots, t_{n-1}) \end{tikzcd}$$

Pushing an pulling the charcater ψ from $\mathbb{A}^1$ to T, we see that for all $t \in T$, the γ-sheaf for the torus $\Phi_{T,\mathfrak{u}_P^\vee,\psi}$ has function $\psi(\mathrm{tr}(t))$, where $\mathrm{tr} : T \to \mathbb{A}^1$ is the standard trace in GL_{n-1}.

Consider the local system $\mathcal{E}$ on Mat_2 obtained by pulling back the Artin-Schreier sheaf $\mathcal{L}_\psi$ along the map $\mathrm{tr} : \mathrm{Mat}_2 \to \mathbb{A}^1$. Since this matches the γ-sheaf $\Phi_{T,\mathfrak{u}_P^\vee,\psi}$ when restricted to T, we find that the γ-sheaf $\Phi_{M,\mathfrak{u}_P^\vee,\psi}$ agrees with $\mathcal{E}$ when we restrict both sheaves to the rss locus of GL_{n-1}. Thus, by the "principal of perverse continuation" [27], we see that $\mathfrak{J}$, the intermediate extension of $\Phi_{M,\mathfrak{u}_P^\vee,\psi}$ to Mat_2, is isomorphic to $\mathcal{E}$ (up to dimension shift and Tate twist). Thus the function J is simply

$$J(m) = \psi(\mathrm{tr}(m)), \tag{16}$$

at least up to a factor of a power of q. Putting together (16) and (15) we find that the Fourier transform is given by:

$$\mathcal{F}_{P^{\mathrm{op}},P}(f)(N) = \sum_{M \in \mathrm{Mat}_{n,n-1}} f(M)\psi(\mathrm{tr}(NM)).$$

But this is simply a standard Fourier transform on vector spaces! Because of this, we see that involutivity holds as a consequence of standard Fourier inversion for vector spaces: $\mathcal{F}_{P,P^{\mathrm{op}}}\mathcal{F}_{P^{\mathrm{op}},P}(f) = q^N f$ for $N = n(n-1)$, for *all* $\mathbb{C}$-valued functions f on the rational points $\overline{G/U(P)}(\mathbb{F}_q)$.[16 17]

6. The Kloosterman Fourier Transform on Affine Quadric Cones

We now describe an involutive Fourier transform for a class of functions on certain affine quadric cones. The involutivity in this case will prove involutivity for two more examples of paraspherical spaces: the Borel for SL_3 and Siegel parabolic for Sp_4.

Let $V \cong k^d$ be a vector space over $\mathbb{F}_q$ of dimension d. We let $\psi : \mathbb{F}_q \to \mathbb{C}$ be a nontrivial additive character, as always. We let

$$X = \{(v, v^\vee) \in V \times V^* : \langle v, v^\vee \rangle = 0\}.$$

Observe that X is an affine quadric cone in k^{2d}, with an isolated conical singularity at the origin. There is a natural scalar action of $\mathbb{G}_m$ on this variety: $\lambda \cdot (v, w) = (\lambda v, \lambda w)$ for $\lambda \in \mathbb{F}_q^\times$.

For $a \in \mathbb{F}_q$ we let

$$\mathrm{Kl}(a) := \sum_{t \in \mathbb{F}_q^\times} \psi\left(\frac{a}{t} + t\right),$$

be the standard Kloosterman function on $\mathbb{F}_q$. Now consider the composite map

$$K : X \times X \to \mathbb{A}^1 \to \mathbb{C}$$

[16]Observe that $\mathcal{F}_{P,P^{\mathrm{op}}}$ will be the Fourier transform with respect to $\overline{\psi}$. This is already seen in the case of SL_2, which is a spacial case of the mirabolic.

[17]Therefore we see that, in the the case of $G = \mathrm{SL}_n$ and P the Mirabolic, the space $\mathcal{S}$ can be extended to the entire space of functions $\mathrm{Fun}(\overline{G/U(P)}(\mathbb{F}_q), \mathbb{C})$. However, $\mathcal{F}$ does clearly preserve the space of special functions (i.e., those whose average is 0 under the scalar action). It seems that the need to restrict to a smaller function space may be related to the presence of singularities in $\overline{G/U(P)}$. As already noted, the mirabolic has an associated paraspherical space that is smooth; this may account for the lack of restrictions on $\mathcal{S}$.

$$((u,u^\vee),(v,v^\vee)) \mapsto \langle u,v^\vee\rangle + \langle v,u^\vee\rangle \mapsto \mathrm{Kl}\left(\langle u,v^\vee\rangle + \langle v,u^\vee\rangle\right).$$

Let

$$\mathcal{S}(X(\mathbb{F}_q),\mathbb{C}) = \left\{ f \in \mathrm{Fun}(X(\mathbb{F}_q),\mathbb{C}) : \sum_{\lambda\in\mathbb{F}_q^\times} f(\lambda x) = 0 \right\}. \tag{17}$$

Then we define on $\mathcal{S}$ the following transform:

$$\mathcal{F}(f)(y) = \sum_{x\in X(\mathbb{F}_q)} f(x)K(x,y). \tag{18}$$

Before we state our theorem, let us observe two things about the space $\mathcal{S}$. Firstly, we may see immediately that $f(0)=0$. Thus $f\in\mathcal{S}$ is solely supported on the smooth locus of the variety X. Secondly, we observe that $\mathcal{F}$ preserves the space $\mathcal{S}$. Indeed:

$$\sum_{\lambda\in\mathbb{F}_q^\times} \mathcal{F}(f)(\lambda y) = \sum_{\lambda\in\mathbb{F}_q^\times}\sum_{x\in X(\mathbb{F}_q)} f(x)K(x,\lambda y).$$

But letting $x=(u,u^\vee)$ and $y=(v,v^\vee)$, we see that $K(x,\lambda y) = \mathrm{Kl}(\langle u,v^\vee\rangle + \langle \lambda v,u^\vee\rangle) = \mathrm{Kl}(\langle \lambda u,v^\vee\rangle + \langle v,\lambda u^\vee\rangle) = K(\lambda x,y)$ by bilinearity. Thus we obtain:

$$\sum_{\lambda\in\mathbb{F}_q^\times}\sum_{x\in X(\mathbb{F}_q)} f(x)K(x,\lambda y) = \sum_{\lambda\in\mathbb{F}_q^\times}\sum_{x\in X(\mathbb{F}_q)} f(x)K(\lambda x,y) = \sum_{x\in X(\mathbb{F}_q)}\sum_{\lambda\in\mathbb{F}_q^\times} f(\lambda^{-1}x)K(x,y) = 0.$$

(As an aside will also note that $\sum_{\lambda\in\mathbb{F}_q^\times}\sum \mathrm{Kl}(x,\lambda y) = 1$, unless $\langle u,v^\vee\rangle + \langle v,u^\vee\rangle = 0$, in which case it is $1-q$.)

We are now ready to state our involutivity theorem:

Theorem 6.1. *The transform $\mathcal{F}$ satisfies*

$$\mathcal{F}^2(f) = q^{2d} f$$

for all $f\in\mathcal{S}(X(\mathbb{F}_q),\mathbb{C})$.

Proof. Observe that the space $\mathcal{S}$ is generated by functions of the form $\delta_x - \delta_{\lambda x}$, where $x=(u,u^\vee)\in X$ and δ_x is the indicator function of $x\in X(\mathbb{F}_q)$. The theorem therefore boils down to verifying the following identity:

$$\sum_{y\in X}\left[K(x,y) - K(\lambda x,y)\right]K(y,z) = \begin{cases} q^{2d} \text{ if } z = x \\ -q^{2d} \text{ if } z=\lambda x \\ 0 \text{ otherwise}\end{cases} \tag{19}$$

□

We will begin by considering the double Fourier transform of the Dirac mass at x:

$$\mathcal{F}^2(\delta_x)(z) = \sum_{y \in X(\mathbb{F}_q)} K(x,y)K(y,z). \tag{20}$$

As above, let $x = (u, u^\vee)$, and $z = (w, w^\vee)$ (where, of course, $\langle u, u^\vee \rangle = \langle w, w^\vee \rangle = 0$). Developing the sum on the left hand side of (6) we obtain:

(21)

$$\sum_{(v,v^\vee) \in X(\mathbb{F}_q)} \left(\sum_{t \in \mathbb{F}_q^\times} \psi\left[t\left(\langle u, v^\vee \rangle + \langle v, u^\vee \rangle\right) + t^{-1}\right] \right) \cdot \left(\sum_{s \in \mathbb{F}_q^\times} \psi\left[s\left(\langle v, w^\vee \rangle + \langle w, v^\vee \rangle\right) + s^{-1}\right] \right)$$

or, swapping the order of the summation:

$$\sum_{t,s \in \mathbb{F}_q^\times} \sum_{(v,v^\vee) \in X(\mathbb{F}_q)} \psi\left[t\left(\langle u, v^\vee \rangle + \langle v, u^\vee \rangle\right)\right] \cdot \psi(t^{-1}) \cdot \psi\left[s\left(\langle v, w^\vee \rangle + \langle w, v^\vee \rangle\right)\right] \cdot \psi(s^{-1}).$$

Now we recall that $X(\mathbb{F}_q)$ is invariant under scaling: thus we may apply the substitution[18] $(v, v^\vee) \mapsto (t^{-1}s^{-1}v, t^{-1}s^{-1}v^\vee)$ for each $t, s \in \mathbb{F}_q^\times$. Then we obtain:

$$\sum_{t,s \in \mathbb{F}_q^\times} \sum_{(v,v^\vee) \in X(\mathbb{F}_q)} \psi\left[s^{-1}\left(\langle u, v^\vee \rangle + \langle v, u^\vee \rangle\right)\right] \cdot \psi(t^{-1}) \cdot \psi\left[t^{-1}\left(\langle v, w^\vee \rangle + \langle w, v^\vee \rangle\right)\right] \cdot \psi(s^{-1})$$

which we may rewrite as:

$$\sum_{t,s \in \mathbb{F}_q^\times} \sum_{(v,v^\vee) \in X(\mathbb{F}_q)} \psi\left[s^{-1}\left(\langle u, v^\vee \rangle + \langle v, u^\vee \rangle + 1\right)\right] \cdot \psi\left[t^{-1}\left(\langle v, w^\vee \rangle + \langle w, v^\vee \rangle + 1\right)\right].$$

Now we may sum with respect to t and s separately. We recall that $\sum_{t \in \mathbb{F}_q^\times} \psi(t^{-1}a)$ is either -1 if $a \neq 0$ or $q-1$ if $a = 0$. So let us define a constructible function $c_{x,z} : X(\mathbb{F}_q) \to \mathbb{C}$ as follows:

$$c_{x,z}(v, v^\vee) = \begin{cases} 1 \text{ if } \langle u, v^\vee \rangle + \langle v, u^\vee \rangle \neq -1 \text{ and } \langle v, w^\vee \rangle + \langle w, v^\vee \rangle \neq -1 \\ -(q-1) \text{ if only one of } \langle u, v^\vee \rangle + \langle v, u^\vee \rangle \text{ or } \langle v, w^\vee \rangle + \langle w, v^\vee \rangle \text{ equals } -1 \\ (q-1)^2 \text{ if } \langle u, v^\vee \rangle + \langle v, u^\vee \rangle = -1 \text{ and } \langle v, w^\vee \rangle + \langle w, v^\vee \rangle = -1. \end{cases}$$

Then we see that

$$\mathcal{F}^2(\delta_x)(z) = \sum_{y \in X(\mathbb{F}_q)} c_{x,z}(y).$$

[18]Applying a substitution that eliminates the intractable Kloosterman sum terms when, as in our situation, we have a product of *two* such Kloosterman sums, bears a striking resemblance to the squaring (followed by polar substitution) technique used to resolve the Gaussian integral.

Geometrically we see that the two constraints $\langle u, v^\vee\rangle + \langle v, u^\vee\rangle = -1$ and $\langle v, w^\vee\rangle + \langle w, v^\vee\rangle = -1$ correspond to affine hyperplanes in $V \times V^*$, while X is a quadric hyperplane in $V \times V^*$. We are thus weighting the points of $X(\mathbb{F}_q)$ by 1, $-(q-1)$, or $(q-1)^2$ depending on whether the point lies in zero, one, or two of these affine hyperplanes.

Let us call the varieties – affine hyperplanes – cut out by $\langle u, v^\vee\rangle + \langle v, u^\vee\rangle = -1$ and $\langle v, w^\vee\rangle + \langle w, v^\vee\rangle = -1$ by Π_1 and Π_2. Observe that replacing x with λx results in $\langle \lambda u, v^\vee\rangle + \langle v, \lambda u^\vee\rangle = -1$; or, equivalently $\langle u, v^\vee\rangle + v, \lambda u^\vee\rangle = -\lambda^{-1}$. This is a parallel translation of Π_1; let us call it $\Pi_1^{+\lambda^{-1}}$. the content of (19) is that the number of $\mathbb{F}_q$-points in $\Pi_1(\mathbb{F}_q) \cap \Pi_2(\mathbb{F}_q) \cap X(\mathbb{F}_q)$ is equal to $\Pi_1^{+\lambda^{-1}}(\mathbb{F}_q) \cap \Pi_2(\mathbb{F}_q) \cap X(\mathbb{F}_q)$ unless $\Pi_1 = \Pi_2$. (If $\Pi_1 = \Pi_2$, then $\Pi_1^{+\lambda^{-1}}$ is parallel, but not equal, to Π_2 – hence the two have empty intersection.)

In any event, we observe that varieties $\Pi_1 \cap \Pi_2 \cap X$ are quadric hypersurfaces in affine space, so we may apply elementary means to count points. We find that:

(22)
$$\sum_{y \in X(\mathbb{F}_q)} c_{x,z}(y) = \begin{cases} q^{2d-1} + q^d - q^{d-1} \text{ if } x = z = 0 \\ q^{2d} - q^{2d-1} + q^d - q^{d-1} \text{ if } x = z, \text{ with neither } 0 \\ -q^{2d-1} + q^d - q^{d-1} \text{ if } x \neq z \text{ are proportional with neither } 0 \\ q^d - q^{d-1} \text{ if either one and only of } x, z = 0; \text{ or else if} \\ \qquad\qquad x \text{ and } z \text{ are not proportional and } \langle u, w^\vee\rangle + \langle w, u^\vee\rangle = 0 \\ -q^{d-1} \text{ if } x \text{ and } z \text{ are not proportional and } \langle u, w^\vee\rangle + \langle w, u^\vee\rangle \neq 0. \end{cases}$$

Observe the pleasant fact that, excepting the ultra-singular point $x = z = 0$, subtracting from a row the row below it yields a single power of q (with sign). Moreover, these strata have been ordered so that each stratum in $X(\mathbb{F}_q)$ lies in the closure of the strata beneath it. For $\lambda \in \mathbb{F}_q^\times$, each stratum is invariant under the action of $(x, z) \mapsto (\lambda x, z)$, except for the $x = z$, $(x, z) \neq (0, 0)$ stratum. If $x = z$, then $(\lambda x, z)$ (for $\lambda \neq 1$) lies in the stratum where $x \neq z$, neither are 0, and x and z are proportional. We conclude that

$$\mathcal{F}^2(\delta_x - \delta_{\lambda x}) = q^{2d}(\delta_x - \delta_{\lambda x}),$$

as claimed.

7. The Case of SL_3

Up to isomorphism, we see that there are two kinds of nontrivial parabolic in G: the Borel (i.e., the 1+1+1-block parabolic), and the 2+1 block parabolic. The latter is the mirabolic; thus we have already covered this case.

7.1. **The Space $\overline{\mathrm{SL}_3/U}$.** We will ow describe the paraspherical space $\overline{G/U(B)}$, for B the standard Borel. As with all Borels, this is the same as the BK-space $\overline{G/[B, B]}$. As we shall see, this case is the smallest-rank example for which $\overline{G/U}$ which is singular.

Observe that, like in the case of SL_2, we have a map

$$G/U(B) \longrightarrow \mathbb{A}^3$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} a \\ d \\ g \end{pmatrix}.$$

However, a simple dimension count shows that G/U has dimension $8 - 3 = 5$, so this map cannot be an isomorphism onto its image.

Observe that, were we to take the *left* quotient by $U(B)$, we would have another map:

$$\begin{gathered} U(B)\backslash G \longrightarrow \mathbb{A}^3 \\ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} g & h & i \end{pmatrix}. \end{gathered} \tag{23}$$

Now we can define a map $G/U(B) \to \mathbb{A}^3$ given by inversion: $gU \mapsto Ug^{-1} \in U(B)\backslash G$. Thus we may postcompose inversion with the map (23) to give us:

$$\begin{gathered} G/U(B) \longrightarrow \mathbb{A}^3 \\ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} dh - ge & gb - ah & ae - db \end{pmatrix}. \end{gathered} \tag{24}$$

In other words, we have two invariants for $G/U(B)$: the first column of g and the third row of g^{-1} (otherwise known as the minors of the third column, or the cross-product of the first two columns; recall that $\det(g) = 1$, which is why there are no denominators). Notice that the canonical pairing of these row-and-column vectors is always 0 (as the product would be the $(3,1)$ entry of $g^{-1}g = \mathrm{Id}$).

Thus, if we let $V = \mathbb{A}^3$ and $V^* = \mathbb{A}^3$, viewed as the dual vector space, we may define the map ϕ:

$$\begin{gathered} G/U(B) \xrightarrow{\phi} V \times V^* \\ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \left(\begin{pmatrix} a \\ d \\ g \end{pmatrix} \right), \begin{pmatrix} dh - ge & gb - ah & ae - db \end{pmatrix}, \end{gathered}$$

whose image is contained in the closed subset $\{(v, v^\vee) \in V \times V^* : \langle v, v^\vee \rangle = 0\}$, and consists of those pairs, subject to this constraint, such that neither v^* nor v is 0. Let

$$X := \{(v, v^\vee) \in V \times V^* : \langle v, v^\vee \rangle = 0\} \tag{25}$$

and

$$X^* := \{(v, v^\vee) \in X : v \neq 0 \text{ and } v^\vee \neq 0\}.$$

Note that X has dimension $6-1=5$, and is affine, while $X-X^*$ is the union of two 3 copies of $\mathbb{A}^3$, glued at the origin. Hence the complement of X^* in X is of codimension 2; thus, by Hartog's Lemma, any regular function on X^* uniquely extends to X. Thus X is the affine closure of X^*. Moreover, by a dimension count, we see that ϕ induces an isomorphism

$$G/U(B) \xrightarrow{\sim} X^*$$

so that

$$\overline{G/U(B)} \cong X.$$

Note that the variety X is precisely the variety over which we have the quadric Kloosterman Fourier transform (for the case $d=3$). We will in fact see the Kloosterman Fourier is precisely the transform we called for by (14). X is a quadric hypersurface in $\mathbb{A}^6$, the affine cone over a quadric hypersurface in $\mathbb{P}^5$. It has an isolated, conical singularity at the origin – since SL_2 and the mirabolics of SL_3 all have smooth paraspherical spaces, we see that this is the smallest example of a singular $\overline{G/U(P)}$.

7.2. **The Slipper Pairing.** Next, we consider the Slipper pairing for opposite Borels:

$$\overline{G/U(B)} \times \overline{G/U(B^{\mathrm{op}})} \to \overline{T}.$$

It is a well-known fact that the double-unipotent invariant functions of SL_n are generated by the determinants of the $i \times i$ square matrices located in the upper left corner of SL_n, $i \leq n-1$ [19; 27].[19] In our case, the two invariants of

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

are a and $ae-db$. These two functions are algebraically independent; thus the Wang monoid in this case is:

$$\overline{T} \cong \mathbb{A}^2 \cong \operatorname{Spec} k[a, ae-bd].$$

Now, we let $X^{\mathrm{op}} = \overline{G/U(B^{\mathrm{op}})} = \{(v, v^\vee) \in V \times V^* : \langle v, v^\vee \rangle = 0\}$, where

$$G/U(B^{\mathrm{op}}) \to X^{\mathrm{op}}$$
$$gU(B^{\mathrm{op}}) \mapsto (\text{third column of } g,\ \text{first row of } g^{-1})$$
$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} \mapsto \left(\begin{pmatrix} c \\ f \\ i \end{pmatrix}, \begin{pmatrix} ei-hf & hc-bi & bf-ec \end{pmatrix} \right).$$

The Slipper pairing is given by

[19]For $i=n$, this is just the determinant of the whole matrix, which is 1 in the SL case. For GL_n this gives another double unipotent invariant. See Grosshans [19], Lemma 18.7, page 104.

$$\mathfrak{S} : X \times X^{\mathrm{op}} \to \mathbb{A}^2$$
$$((v, v), (w, w^\vee)) \mapsto (\langle w, v^\vee \rangle, \langle v, w^\vee \rangle),$$

since the upper right entry of $h^{-1}g$ is given by (first row of h^{-1}) $\cdot$ (first column of g), while the determinant of the upper right 2×2 is the same as the $(3, 3)$-entry of $g^{-1}h$, which is (third row of g^{-1}) $\cdot$ (third column of h). As we would expect, the pairing on the "G-diagonal", in other words, $\mathfrak{S}(gU(B), gU(B^{\mathrm{op}}))$, is always $(1, 1)$, the identity of the monoid $\mathbb{A}^2$.

7.3. **The Function** J**.** We now examine the function J on the Wang monoid $\mathbb{A}^2$. To accomplish this we must analyze the adjoint representation of $M^\vee = T^\vee$ on $\mathfrak{u}^\vee$. $X^*(T^\vee) = X_*(T) = \{(a_{11} + a_2\lambda_2 + a_3\lambda_3 : a_1 + a_2 + a_3 = 0\}$. Recall that $\mathrm{SL}_3^\vee = \mathrm{PSL}_3$. We see that the the adjoint action of $T^\vee$ in $\mathfrak{u}^\vee$ has weights $\lambda_1 - \lambda_2$, $\lambda_2 - \lambda_3$, $\lambda_1 - \lambda_3$ (i.e.,, the positive coroots of SL_3. Hence the Kloosterman sheaf (forgetting the Tate twist) on the torus T of SL_3 is given by the pull-push of the Artin-Schreier sheaf along the map:

$$\begin{tikzcd}
& \mathbb{G}_m^3 \arrow[dl, "\mathrm{Tr}"'] \arrow[dr, "p_{\underline{\lambda}}"] & \\
\mathbb{A}^1 & (t_1, t_2, t_3) \arrow[dl, mapsto] \arrow[dr, mapsto] & T \\
t_1 + t_2 + t_3 & & \mathrm{diag}(t_1t_3, t_2t_1^{-1}, t_2^{-1}t_2^{-1})
\end{tikzcd}$$

Now, recall that our two invariants of $\overline{T}$ are the upper left corner, and the determinant of the upper left 2×2. Let us call these invariants A and B, respectively. We find that $A(\mathrm{diag}(t_1t_3, t_2t_1^{-1}t_2^{-1}t_2^{-1})) = t_1t_3$ and $B(\mathrm{diag}(t_1t_3, t_2t_1^{-1}t_2^{-1}t_2^{-1})) = t_2t_3$. Thus we find that $A + B = t_3(t_1 + t_2)$, and that $t_1 + t_2 + t_3 = (A + B)t_3^{-1} + t_3$.

Therefore, as a function on T, we may write

$$p_{\underline{\lambda}!}\mathrm{Tr}^*(\psi)(A, B) = \sum_{t_3 \in \mathbb{F}_q^\times} \psi((A + B)t_3^{-1} + t_3) = \mathrm{Kl}(A + B).$$

for $A, B \in \mathbb{F}_q^\times$.

Once more, we may apply the perverse continuation principal to see that J extends to the function $\mathrm{Kl}(A + B)$ on the entire Wang Monoid $\overline{T} = \mathbb{A}^2$.

Therefore the Fourier transform is given by

$$\mathcal{F}(f)(y) = \sum_{x \in X(\mathbb{F}_q)} f(x)K(x, y) \tag{26}$$

where $K(x, y) = \mathrm{Kl}\left(\langle u, v^\vee \rangle + \langle v, u^\vee \rangle\right)$ (adopting as before, the notation $x = (u, u^\vee)$ and $y = (v, v^\vee)$).

Now we examine the notion of a special function for $\overline{\mathrm{SL}_3/U}$. In this case, the set of coroots whose 1-parameter subgroup in $U^\vee \subset \mathrm{PGL}_3 = G^\vee$ are simply the set of positive coroots (under the standard upper-triangular convention). Thus $\Pi_B^\vee$ is simply equal to the set of the two standard simple coroots, $t \mapsto \mathrm{diag}(t, t^{-1}, 1)$ and $t \mapsto \mathrm{diag}(1, t, t^{-1})$. Their sum is thus the coroot $\alpha_B^\vee : t \mapsto \mathrm{diag}(t, 1, t^{-1})$. Recall that we must think of this as giving a right $\mathbb{G}_m$ action on SL_3/U. We see that is scales the first column of a matrix by t, and, similarly, it scales the third row of the inverse by t. Thus the action of $\mathbb{G}_m$ is given by scaling $(v, v) \mapsto (tv, tv^\vee)$ for $t \in \mathbb{F}_q^\times$ and $(v, v^\vee) \in X(\mathbb{F}_q)$, and our space our space of special functions are precisely those that satisfy (17).

We conclude that the Foruier transform for SL_3 and opposite Borels is precisely the Fourier transform (18) for a 3-dimensional space V. Thus the involutivity property follows from Theorem 6.1.

7.4. **Normalized Intertwining Operators for** $\mathrm{SL}_3(\mathbb{F}_q)$**.** We will now discuss how our Fourier transform for opposite Borels in combination with the standard Braverman-Kazhdan Fourier transform for adjacent Borels give rise to normalized intertwining operators on $\mathrm{SL}_3(\mathbb{F}_q)$.

7.4.1. *Review of Braverman-Kazhdan's Normnalized Intertwiners for Borels.* We first briefly review the definition of a normalized inertwining operator. Given a fixed Levi factor $M \subset G$, we let $\mathcal{P}_M$ denote the collection of all parabolics P, with Levi quotient isomorphic to M, containing M. As always, if $P_i \in \mathcal{P}_M$, we let $U_i = R_u(P_i)$ denote its unipotent radical. We would like to find transforms between function spaces

$$\mathcal{F}_{ji} : \mathcal{S}(\overline{G/U_i}, \mathbb{C}) \to \mathcal{S}(\overline{G/U_j}, \mathbb{C})$$

intertwining the natural $G \times M$-actions, such that

$$\mathcal{F}_{ii} = \mathrm{Id} \text{ and } \mathcal{F}_{kj} \circ \mathcal{F}_{ji} = \mathcal{F}_{ki}.$$

For $G = \mathrm{SL}_3$ and $M = T$ the standard torus, there are six Borels B containing T forming a torsor under the Weyl group of SL_3 (i.e., the symmetric group S_3). If we let B be the standard (upper triangular) Borel, then we let $B_w = n_w B n_w^{-1}$, where $n_w \in N(T)$ is a lift of w. We then have a fixed bijection between the W and $\mathcal{P}_T$. We write simply $\mathcal{F}_{w',w}$ for the intertwiner:

$$\mathcal{S}(\overline{G/U_w}, \mathbb{C}) \to \mathcal{S}(\overline{G/U_{w'}}, \mathbb{C}).$$

Let us recall how Braverman-Kazhdan construct normalized intertwining operators for Borels in [6]. This is done over local fields; we let k be a local field, and $\psi : k \to \mathbb{C}$ a fixed additive character.

The idea, which seems to have originated in the works of Gelfand-Graev (see footnote 2) and Kazhdan [20], is to exploit the geometry of G/B and G/U_{s_α} for s_α a simple reflection. In particular, it turns out that G/U and G/U_{s_α} are naturally-dual rank-2 vector bundles (minus their 0-sections) over a common base-space. This linear structure in turn us allows us to write a transformation $\mathcal{F}_{B_{s_\alpha}, B}$ as a classical Fourier transform on each fiber. The involutivity property then follows straightforwardly from Fourier inversion for $\mathbb{A}^2$ of the kind we have already seen.

Let $B' = B_s$, and $U' = U_{s_\alpha}$. Then $U \cap (U')^{\mathrm{op}}$ is a single 1-parameter subgroup associated to the root α; we call[20] this $u_\alpha : \mathbb{G}_a \to G$. We assume G is simply connected, so that u_α and $u_{-\alpha}$ generate a subgroup isomorphic to SL_2 in G.

Let Q denote the subgroup of G generated by U and U'. Note that $Q = [P_\alpha, P_\alpha]$, the commutator group of the minimal non-Borel parabolic associated to α. Then we notice that:

$$\begin{tikzcd} G/U \arrow[dr] & & G/U' \arrow[dl] \\ & G/Q & \end{tikzcd}$$

exhibits G/U as a $Q/U \simeq SL_2/\mathcal{U} \simeq \mathbb{A}^2 \setminus \{(0,0)\}$ fibration over G/Q. Likewise, G/U' is a $Q/U' \simeq SL_2/\mathcal{U}^{\mathrm{op}} \simeq \mathbb{A}^2 \setminus \{(0,0)\}$ fibration over G/Q. (Here $\mathcal{U}$ refers to the upper-unipotent subgroup of SL_2, while $\mathcal{U}^{\mathrm{op}}$ refers to the lower-unipotent subgroup of SL_2; the SL_2 here refers to the the subgroup of Q generated by u_α and $u_{-\alpha}$.) In other words, both G/U and G/U' are rank two vector bundles over G/Q, with zero section removed.

Let $\widetilde{G/U}$ and $\widetilde{G/U'}$ denote the corresponding rank two vector bundles over G/Q; equivalently, these are the *relative* affine closures of G/U and G/U' over G/Q. There is a fiberwise duality between these two rank-2 vector bundles; in fact, there is a G-invariant form

$$\langle -, - \rangle : \widetilde{G/U} \times_{G/Q} \widetilde{G/U'} \to \mathbb{A}^1 \tag{27}$$

which we call the *Braverman-Kazhdan Pairing*. (Note that this is not the pair of Slipper's.) The BK pairing reduces to the pairing we have already on $\overline{\mathrm{SL}_2/U}$ and $\overline{\mathrm{SL}_2/U^{\mathrm{op}}}$ in (6). Indeed, let $\phi : \mathrm{SL}_2 \to G$, where

$$\phi\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = u_\alpha(x); \quad \phi\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = u_{-\alpha}(x); \quad \phi\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \alpha^\vee(t).$$

Then we observe that if gU and hU' both lie over the same point in G/Q, we have $U'h^{-1}gU$ is a well-defined element of $\phi(u_{-\alpha})\backslash\phi(\mathrm{SL}_2)/\phi(u_\alpha)$. Of course, this last double quotient is isomorphic (via ϕ^{-1}) to

$$\mathcal{U}^{\mathrm{op}}\backslash SL_2/\mathcal{U} \cong \mathbb{A}^1$$

via (5). As in the SL_2 case, this pairing is clearly G-invariant since $U'h^{-1}gU = U'(g'h)^{-1}(g'g)U$.

Now, we define:

$$C(\widetilde{G/U}(k), \mathbb{C}) \xrightarrow{\mathcal{F}_{B,B'}} C(\widetilde{G/U'}(k), \mathbb{C})$$

[20] We will be a bit cavalier and sometimes refer to u_α as a subgroup of G; when being precise, we will mean a particular map $\mathbb{G}_a \to G$.

as the fiberwise Fourier transform with respect to ψ and the above-defined vector-space duality. That is to say, we "pull-push" functions along the upper roof of the Cartesian diagram:

$$
\begin{tikzcd}
& \widetilde{G/U'} \times_{G/Q} \widetilde{G/U} \arrow[dl] \arrow[dr] & \\
\widetilde{G/U'} \arrow[dr] & & \widetilde{G/U} \arrow[dl] \\
& G/Q &
\end{tikzcd}
\tag{28}
$$

with respect to the kernel $\psi\left(\langle -, -\rangle\right)$.

In formulae, let $f \in C(\widetilde{G/U}(k), \mathbb{C})$. We pull f back to a right U-invariant function on G, which we, somewhat abusively, also call f. Then, for all $g \in G$, and $\beta, \delta \in k$, we have:

$$
\left(\mathcal{F}_{B^{\mathrm{op}},B}(f)\right)\left[g\phi\begin{pmatrix} * & \beta \\ * & \delta \end{pmatrix}\right] = \int_{\mathbb{A}^2} f\left(g\phi\begin{pmatrix} \alpha & * \\ \gamma & * \end{pmatrix}\right)\psi\left(\alpha\delta - \beta\gamma\right) d\alpha d\gamma,
$$

where $*$ means arbitrary entries such that $\begin{pmatrix} * & \beta \\ * & \delta \end{pmatrix}, \begin{pmatrix} \alpha & * \\ \gamma & * \end{pmatrix} \in \mathrm{SL}_2(k)$. (The output is independent of the choice of $*$'s because f is right $U(B) \supset u_\alpha$-invariant, and $\mathcal{F}_{B^{\mathrm{op}},B}(f)$ is right $U(B') \supset u_{-\alpha}$-invariant.)

Once the transform has been established of B and B' separated by a simple reflection, the general $\mathcal{F}_{B',B}$ is constructed as a composition of such "simple" Fourier transforms. One must verify, of course, that the resulting transform $\mathcal{F}_{B_w,B}$ is independent of the decomposition of w into simple reflections. For formal reasons related to generators and relations for Coxeter groups, this reduces to verifying that

$$
\mathcal{F}_{e,s_\alpha} \circ \mathcal{F}_{s_\alpha,e} = \mathcal{F}_{e,e} = \mathrm{Id} \tag{29}
$$

and

$$
\mathcal{F}_{w'w,w} \circ \mathcal{F}_{w,e} = \mathcal{F}_{w'w,e} \tag{30}
$$

in the case in which $\ell(w'w) = \ell(w') + \ell(w)$, where ℓ is the length function on the Weyl group. Property (29) is involutivity, which follows from Fourier inversion for $\mathbb{A}^2$; (30) follows, essentially, from Fubini's theorem.

7.4.2. *Finite Field Subtleties.* If we attempt to repeat this construction in the context of finite fields, we quickly bump up against an obstruction. We may perfectly well pull-push functions on rational points along (28); this will define an involutive Fourier transform $\mathcal{F}_{s_\alpha,e}$.

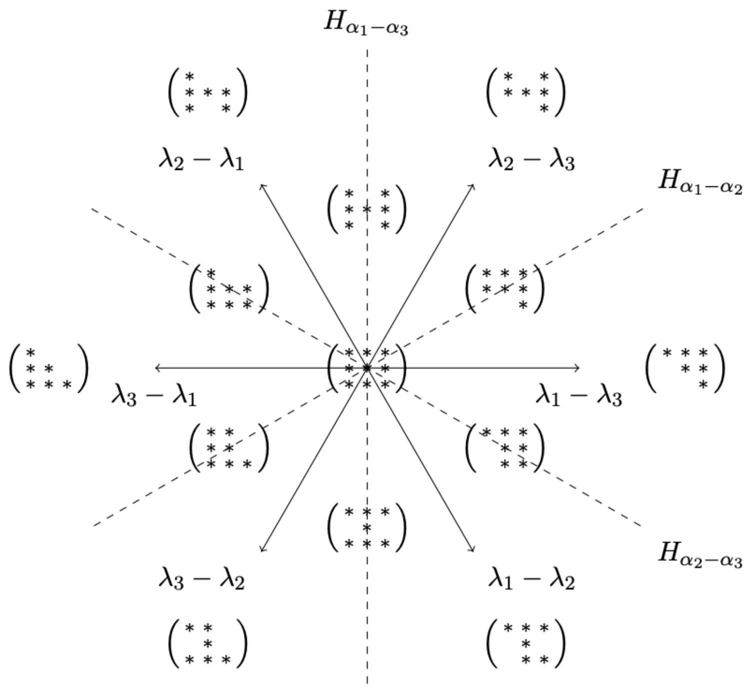
But we can proceed no further in composing this transform by *another* such transform[21] $\mathcal{F}_{s_\alpha s_\beta, s_\alpha}$: applying the transform $\mathcal{F}_{s_\alpha, e}$ results in a function on $\widetilde{G/U'}(\mathbb{F}_q)$, while the domain of $\mathcal{F}_{s_\alpha s_\beta, s_\alpha}$ consists of the set of functions on a *different* relative affine closure of G/U'. Of course, over local fields, these discrepancies are irrelevant because the difference between $\widetilde{G/U}$ and G/U is of measure 0, so we may restrict to functions on G/U everywhere. But, as we have discussed, Zariski closed subsets of varieties over finite fields have positive measure; we cannot simply ignore them.

There are two ways we might attempt to circumvent this problem. One, which we explore in this section, is to restrict our space of functions to those functions f such that the support of $\mathcal{F}_{s_\alpha, e}(f)$ is entirely in $G/U'(\mathbb{F}_q) \subset \widetilde{G/U'}(\mathbb{F}_q)$. Another is to try to extend the transform to functions on the absolute affine closure $\overline{G/U}$ (which contains all the different $\widetilde{G/U}$ in a canonical way).

For now, let us try to understand the first approach in the case of SL_3. We will consider all of our functions to have domain equal to the full affine closure $\overline{G/U}(\mathbb{F}_q)$. But we will constrain the functions' support so that the composition of the "simple" Braverman-Kazhdan transforms $\mathcal{F}_{ws_\alpha, w}$ is always well-defined. We will discover that our "opposite" Fourier transform (43), when restricted to this function space, does indeed agree with the corresponding composition of simple Braverman-Kazhdan Fourier transforms.

7.4.3. *The Geometry of Braverman-Kazhdan Pairings for* SL_3. We begin by reproducing Makisumi's image the spherical apartment for SL_3, as a visual aid. Recall that each facet corresponds to a different standard parabolic; these parabolics have been superimposed. The open 60-degree cones (the Weyl chambers) correspond to Borels, and Borels aree adjacent if the corresponding Weyl chambers share a wall:

[21] Observe that while B is adjacent to B_{s_α} for a simple reflection s_α, it is no longer true that B_{s_α} is adjacent to $B_{s_\beta s_\alpha}$ for s_β another simple reflection. Rather, we see that we must conjugate s_β by s_α to produce simple reflection with respect to the Borel B_{s_α}. Hence we see that $B_{(s_\alpha s_\beta s_\alpha)s_\alpha} = B_{s_\alpha s_\beta}$ is the desired adjacent Borel. Notice that the order of α and β is flipped from what one might naively expect.



The Spherical Apartment and Coroots for SL_3

Now we will examine the Braverman-Kazhdan pairing for adjacent Borels in SL_3. Let B' be the Borel corresponding to the Weyl chamber containing $\lambda_2-\lambda_3$ in the illsutration above. Let $X' = \overline{G/U(B')}$. We see that $X' = \{(v^*, v) \in V^* \times V : \langle v^*, v\rangle = 0\}$, with the map

$$G/U(B') \to X'$$
$$gU(B') \mapsto (\text{row 3 of } g^{-1},\ \text{column 2 of } g)$$
$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 & 0 & * \\ * & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \mapsto \left(\begin{pmatrix} dh-ge & gb-ah & ae-db \end{pmatrix}, \begin{pmatrix} b \\ e \\ h \end{pmatrix}\right).$$

Now both $G/U(B)$ and $G/U(B')$ have natural maps to $G/[P,P]$, where P is the 2+1 parabolic (in fact, $[P,P]$ is the subgroup of G generated by $U(B)$ and $U(B')$). Thus there are maps between the corresponding affine closures; these are given by

$$X \to \mathbb{A}^3$$
$$(v, v^\vee) \mapsto v^\vee$$

and

$$X' \to \mathbb{A}^3$$
$$(v, v^\vee) \mapsto v^\vee.$$

Hence we may consider the varieties

$$\overline{G/U(B)} \times_{\overline{G/[P,P]}} \overline{G/U(B')} = \{(v, v^\vee, w) \in V \times V^* \times V : \langle v, v^\vee \rangle = 0 \text{ and } \langle w, v^\vee \rangle = 0\}, \tag{31}$$

and

$$\widetilde{G/U(B)} \times_{G/[P,P]} \widetilde{G/U(B')} = \{(v, v^\vee, w) \in V \times V^* \times V : \langle v, v^\vee \rangle = 0,\ \langle w, v^\vee \rangle = 0,\ v^\vee \neq 0\}. \tag{32}$$

This latter space is a product of two rank-two vector bundles over $G/[P,P]$, and it possesses the Braverman-Kazhdan pairing

$$\langle -, - \rangle : \widetilde{G/U(B)} \times_{G/[P,P]} \widetilde{G/U(B')} \to \mathbb{A}^1. \tag{33}$$

Explicitly, this is given as follows. Observe that $v \wedge w \in \bigwedge^2(V) \cong V^*$. (We make this isomorphism canonical by insisting that $\bigwedge^3 V$ has chosen basis $e_1 \wedge e_2 \wedge e_3$, giving us a specified isomorphism $\bigwedge^3 V \cong \mathbb{A}^1$. Here the e_i are the standard basis vectors of $V = \mathbb{A}^3$.) Since v and w both lie in $\ker(v^\vee)$, we see that $v \wedge w = \lambda v^\vee$. Then the map $(v, , v^\vee, w) \mapsto \lambda$ defines the map (33). When we identify all the vector spaces as k^3 with the standard dot product representing vector-covector evaluation, we may write the Braverman-Kazhdan pairing in formulae as:

$$(v, v^\vee, w) \mapsto \frac{v \times w}{v^\vee}$$

where $\times$ means the classical 3-dimensional cross product. The ratio is meaningful because $v \times w$ and $v^\vee$ are, as we have seen, proportional.

Observe that the Braverman-Kazhdan Pairing (33) does *not* algebraically extend to (31): the ratio λ cannot be defined if $v^\vee = 0$. This is one reason that extending even the "adjacent" Braverman-Kazhdan transformation to all functions on $\overline{G/U}(\mathbb{F}_q)$ is not so straightforward.

7.4.4. *Restricted Function Spaces and Composition of Transforms.* We want to constrain our functions f so that we may compose Braverman-Kazhdan transforms.

Firstly: we observe that the Braverman-Kazhdan transform corresponding the the flip across the $H_{\alpha_1-\alpha_2}$ hyperplane in the spherical apartment, is defined on our model of $\overline{\mathrm{SL}_3/U}$ by:

$$\mathcal{F}_{\mathrm{BK}}(f)(v, v^\vee) = \sum_{u \in (v^\vee)^\perp} f(u, v^\vee) \psi \left(\frac{u \times v}{v^\vee} \right).$$

We see that we must have

$$f(v, 0) = 0 \tag{34}$$

if the transform is even to be well-defined. (Setting $f(v,0)=0$ allows us to ignore the terms where we would have a division by 0 problem.) Similarly, if we want to be able to define the BK Fourier transform corresponding to the flip about $H_{\alpha_2-\alpha_3}$, we will need to have

$$f(0,v^\vee)=0 \tag{35}$$

for all $v^\vee$. Observe that these two constraints mean that f is entirely supported on $G/U \subset \overline{G/U}$.

Next we demand that $\mathcal{F}_{\mathrm{BK}}(f)$ vanish on the set $\{(v,v^\vee) : v=0\}$; we see that:

$$0=\mathcal{F}_{\mathrm{BK}}(f)(0,v^\vee)=\sum_{u\in(v^\vee)^\perp} f(u,v^\vee)\psi\left(\frac{u\times 0}{v^\vee}\right)=\sum_{u\in(v^\vee)^\perp} f(u,v^\vee).$$

Hence

$$\sum_{u\in(v^\vee)^\perp} f(u,v^\vee)=0 \tag{36}$$

for all $v^\vee$. Likewise, considering the $H_{\alpha_2-\alpha_3}$ flip, we see that we want

$$\sum_{v^\vee\in u^\perp} f(u,v^\vee)=0 \tag{37}$$

for all u.

We find that the most natural constraint, in the spirit of (17) and our notion of "special" functions (see footnote 13, in particular), is as follows:

$$\begin{aligned}\mathcal{S}'(X(\mathbb{F}_q),\mathbb{C})=\Bigg\{f : \forall\, v,v^\vee \text{ with } \langle v,v^\vee\rangle=0, \sum_{\lambda\in\mathbb{F}_q^\times} f(\lambda v,v^\vee)=0,\\ \sum_{\lambda\in\mathbb{F}_q^\times} f(v,\lambda v^\vee)=0, \text{ and } \sum_{\lambda\in\mathbb{F}_q^\times} f(\lambda v,\lambda v^\vee)=0\Bigg\}.\end{aligned} \tag{38}$$

Observe that this automatically implies (34) and (35). Moreover, we see that these constraints also imply (36) and (37). In fact, the constraints impose by $\mathcal{S}'$ are somewhat stronger than those given by (34), (35), (36) and (37). It is possible we may be over-constraining $\mathcal{S}'$; however, we will eventually use the full force of the constraints on $\mathcal{S}'$ in comparing compositions of BK transforms to the Kloosterman transform. Finally, observe that $\mathcal{F}_{\mathrm{BK}}$ does indeed preserve the space $\mathcal{S}'$. For this all these reasons, we have no obstruction to composing BK transforms.[22]

[22]Following the notion of specialness for a $P \rightsquigarrow P'$ intertwiner, as described in footnote 13, we see that these three toroidal averaging constraints correspond to exactly to being special for each of the intertwiners $B \rightsquigarrow B'$ we will consider below.

We now wish to compose several of these BK transforms and compare it with our Kloosterman transform for opposite Borels. To simplify notation, we will let c_i represent the ith column vector of a matrix g in SL_3, and let $\overline{r}_i$ represent the ith row vector of g^{-1}. We will compose the three flips along the upper 180-degree arc in the spherical apartment illustrated above, going from the upper-triangular Borel to the lower-triangular. We first tabulate the invariants for each G/U_w we will encounter en route:

$\mathrm{SL}_3 \Big/ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$,	$\mathrm{SL}_3 \Big/ \begin{pmatrix} 1 & 0 & * \\ * & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$,	$\mathrm{SL}_3 \Big/ \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & * \\ * & 0 & 1 \end{pmatrix}$,	$\mathrm{SL}_3 \Big/ \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix}$
c_1	c_2	c_2	c_3
$\overline{r_3}$	$\overline{r_3}$	$\overline{r_1}$	$\overline{r_1}$

Therefore, we may write the composite transform as:

$$\mathcal{F}_{\mathrm{comp}}(f)(c_3,\overline{r_1}) = \sum_{\substack{c_2 \\ c_2\cdot\overline{r_1}=0}} \sum_{\substack{\overline{r_3} \\ c_2\cdot\overline{r_3}=0}} \sum_{\substack{c_1 \\ c_1\cdot\overline{r_3}=0}} f(c_1,\overline{r_3})\psi\left(\frac{c_1\times c_2}{\overline{r_3}}\right)\psi\left(\frac{\overline{r_3}\times\overline{r_1}}{c_2}\right)\psi\left(\frac{c_2\times c_3}{\overline{r_1}}\right), \tag{39}$$

where the vectors over which we are summing are presumed to be nonzero. We want to compare this with the Kloosterman transformation:

$$\mathcal{F}(f)(c_3,\overline{r_1}) := \sum_{(c_1,\overline{r_3})\in X(\mathbb{F}_q)} f(c_1,\overline{r_3})K[(c_1,\overline{r_3}),(c_3,\overline{r_1})] \tag{40}$$

where $K[(c_1,\overline{r_3}),(c_3,\overline{r_1})] := \mathrm{Kl}(c_1\cdot\overline{r_1}+c_3\cdot\overline{r_3})$.

Proposition 7.1. *For all $f\in\mathcal{S}'$, we have*

$$\mathcal{F}_{comp}(f) = \mathcal{F}(f).$$

Proof. If we fix $c_1,\overline{r_3}$ (and of course c_3 and $\overline{r_1}$ are fixed to begin with!), we may find the total coefficient of $f(c_1,\overline{r_3})$ in the sum (39):

$$\sum_{\substack{c_2 \\ c_2\in\overline{r_3}^{\perp}\cap\overline{r_1}^{\perp}}} \psi\left(\frac{c_1\times c_2}{\overline{r_3}}\right)\psi\left(\frac{\overline{r_3}\times\overline{r_1}}{c_2}\right)\psi\left(\frac{c_2\times c_3}{\overline{r_1}}\right).$$

If $\overline{r_3}$ and $\overline{r_1}$ are not proportional, then c_2 lives in the 1-dimensional subspace $\overline{r_1}^{\perp}\cap r_3^{\perp}$: in fact, it is precisely the subspace generated by $\overline{r_3}\times\overline{r_1}$. Let $c_2=\lambda\cdot\overline{r_3}\times\overline{r_1}$, and recall Lagrange's formula, $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$. We find that $c_1\times c_2=\lambda(c_1\cdot\overline{r_1})$, since we assume that $c_1\cdot\overline{r_3}=0$. Likewise, $c_2\times c_3=\lambda(c_3\cdot\overline{r_3})$. We manifestly have $\frac{\overline{r_3}\times\overline{r_1}}{c_2}=\lambda^{-1}$, so we find that the coefficient of $f(c_1,\overline{r_3})$ is:

$$\sum_{\lambda\in\mathbb{F}_q^\times}\psi([c_1\cdot\overline{r_1}+c_3\cdot\overline{r_3}]\lambda+\lambda^{-1}) = \mathrm{Kl}(c_1\cdot\overline{r_1}+c_3\cdot\overline{r_3}).$$

This is precisely what we want: we see that, "generically", the transform $\mathcal{F}_{\text{comp}}$ resulting from the composite of the three BK transforms gives the same kernel as our Kloosterman Fourier transform. More precisely, if $\overline{r_3}$ and $\overline{r_1}$ are independent, then the total coefficient of $f(c_1, \overline{r_3})$ in the expansion of $\mathcal{F}_{\text{comp}}(f)$ in (39) is exactly $K((c_1, \overline{r_3}), (c_3, r_1)$, the coefficient in its expansion in (40). Observe that independence of $\overline{r_3}$ and $\overline{r_1}$ is an "open" or "generic" condition.

We have shown that, after amalgamating all the summands with a factor of $f(c_1, \overline{r_3})$ in (39), we will obtain a sum that "generically" matches (40) term-wise. Now we must examine the remaining "boundary" terms. We will show that the two "boundary" sums agree (though only "in totality" – not necessarily term-wise). More precisely, we will show that the sum of all the terms in (39) and (40) for which $\overline{r_3}$ and $\overline{r_1}$ are dependent is, in both cases, 0. This is where we make critical use of the assumption that f belongs to $\mathcal{S}'$.

Let us consider the remaining "boundary terms" of the sum (39). In this case, $\overline{r_1}$ and $\overline{r_3}$ are proportional, and c_2 lives in the 2-dimensional space $\overline{r_1}^{\perp} = \overline{r_3}^{\perp}$. Let $\overline{r_3} = \mu\overline{r_1}$. We see that the sum of such terms is:

$$\sum_{\substack{c_2 \\ c_2 \cdot \overline{r_1} = 0}} \sum_{\mu \in \mathbb{F}_q^\times} \sum_{\substack{c_1 \\ c_1 \cdot \overline{r_1} = 0}} f(c_1, \mu\overline{r_1}) \psi\left(\frac{c_1 \times c_2}{\mu\overline{r_1}}\right) \psi\left(\frac{\mu\overline{r_1} \times \overline{r_1}}{c_2}\right) \psi\left(\frac{c_2 \times c_3}{\overline{r_1}}\right)$$

The second ψ factor disappears because $\overline{r_1} \times \overline{r_1} \ = 0$. And now we may apply the substitution $c_1 \to \mu c_1$, which gives us

$$\sum_{\substack{c_2 \\ c_2 \cdot \overline{r_1} = 0}} \sum_{\mu \in \mathbb{F}_q^\times} \sum_{\substack{c_1 \\ c_1 \cdot \overline{r_1} = 0}} f(\mu c_1, \mu\overline{r_1}) \psi\left(\frac{c_1 \times c_2}{\overline{r_1}}\right) \psi\left(\frac{c_2 \times c_3}{\overline{r_1}}\right)$$

Since the last two factors are independent of μ, we find the whole expression is 0 because $f \in \mathcal{S}'$. Hence for $f \in \mathcal{S}'$, the total sum of those terms in (39) for which $\overline{r_3}$ is proportional to $\overline{r_1}$ is 0.

Now we examine the total sum of those terms in (40) for which $\overline{r_3}$ is proportional to $\overline{r_1}$. This is:

$$\sum_{c_1 \in \overline{r_1}^{\perp}} \sum_{\overline{r_3} = \mu\overline{r_1}} f(c_1, \overline{r_3}) K[(c_1, \overline{r_3}), (c_3, \overline{r_1})] = \sum_{c_1 \in \overline{r_1}^{\perp}} \sum_{\lambda, \mu \in \mathbb{F}_q^\times} f(c_1, \mu\overline{r_1}) \psi[\lambda(c_1 \cdot \overline{r_1} + \mu c_3 \cdot \overline{r_1}) + \lambda^{-1}].$$

But $c_3 \cdot \overline{r_1} = 0$ by assumption, whence the above becomes:

$$\sum_{c_1 \in \overline{r_1}^{\perp}} \sum_{\mu \in \mathbb{F}_q^\times} f(c_1, \mu\overline{r_1}) \cdot \mathrm{Kl}(c_1 \cdot \overline{r_1}) = 0,$$

since $f \in \mathcal{S}'$.

Thus the total sum of those terms in (40) for which $\overline{r_3}$ is proportional to $\overline{r_1}$ is 0, and so agrees with the total sum of such terms in (39). Since we have already matched up the other terms in the two summations, the proposition follows.

$\square$

Putting Proposition 7.1 together with the involutivities of $\mathcal{F}_{BK}$ and our Kloosterman $\mathcal{F}$, we may now state the following theorem:

Theorem 7.1. *Let $G = \mathrm{SL}_3$, T the standard torus. Let $\mathcal{B}_T$ denote the set of (six) Borels of G cotaining T. For each Borel $B \in \mathcal{B}_T$, let $(\Phi^\vee)^+_B$ denote the collection of positive coroots for B. For each coroot $\alpha^\vee$, let $T_{\alpha^\vee}$ denote the corresponding 1-dimensional subtorus. Let us define the space*

$$\mathcal{S}'(\overline{SL_3/U(B)}(\mathbb{F}_q), \mathbb{C})$$

as the collection of $\mathbb{C}$-valued functions on $\overline{G/U(B)}(\mathbb{F}_q)$, whose average under the right action of $T_{\alpha^\vee}$ is 0 for each $\alpha^\vee \in (\Phi^\vee)^+_B$. Then the transforms $q^{-1}\mathcal{F}_{BK}$ between adjacent Borels generate a family of normalized intertwining operators.[23] *Moreover, the intertwiner for opposite Borels agrees with $q^{-3}\mathcal{F}$, where $\mathcal{F}$ is defined as in (14).*

Viewing everything as functions on the set $X(\mathbb{F}_q)$, for the variety

$$X = \{(v, v^\vee) \in V \times V^* : \langle v, v^\vee \rangle = 0\},$$

where $V = \mathbb{A}^3$, we may say that there exists a G-equivariant action of $S_3 = W_T(\mathrm{SL}_3)$ on the space $\mathcal{S}'(X(\mathbb{F}_q), \mathbb{C})$ as defined in (38). This action is G-equivariant, and T-equivariant where the right T-action twisted by W. It is generated by the two transformations:

$$\mathcal{F}_{\alpha_1-\alpha_2}(f)(v, v^\vee) = q^{-1} \sum_{u \in (v^\vee)^\perp} f(u, v^\vee)\psi\left(\frac{u \times v}{v^\vee}\right) \tag{41}$$

and

$$\mathcal{F}_{\alpha_2-\alpha_3}(f)(v, v^\vee) = q^{-1} \sum_{u^\vee \in v^\perp} f(v, u^\vee)\psi\left(\frac{u^\vee \times v^\vee}{v}\right) \tag{42}$$

while the longest element of W acts via the quadratic Kloosterman Fourier transform:

$$\mathcal{F}_{w_0}(f)(v, v^\vee) = q^{-3} \sum_{(u,u^\vee) \in X(\mathbb{F}_q)} f(u, u^\vee) K[(u, u^\vee), (v, v^\vee)] \tag{43}$$

where $K(x, y) = \mathrm{Kl}\left(\langle u, v^\vee \rangle + \langle v, u^\vee \rangle\right)$.

8. The case of Sp_4

In this section, we let $G = \mathrm{Sp}_4$.

[23] The factor of q^{-1} is to make the transform truly involutive.

8.1. **Conventions.** We will adopt the convention in Makisumi:

$$\mathrm{Sp}_4 = \left\{ g \in \mathrm{GL}_4 \,:\, g^{\mathrm{T}} \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix} g = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix} \right\}.$$

This slightly unusual convention is superior to the usual $\begin{pmatrix} 0 & \mathrm{Id}_2 \\ -\mathrm{Id}_2 & 0 \end{pmatrix}$ insofar as it permits us to let the standard Borel be given by upper triangular matrices. The maximal torus is given by matrices of the form:

$$\begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix}$$

The projections α_1 and $\alpha_2 : T \to \mathbb{G}_m$ give us a system of fundamental weights for Sp_4; the set of roots is:

$$\Phi = \{\pm 2\alpha_1,\, \pm 2\alpha_2,\, \pm\alpha_1 \pm \alpha_2\}$$

while the set of coroots is given by:

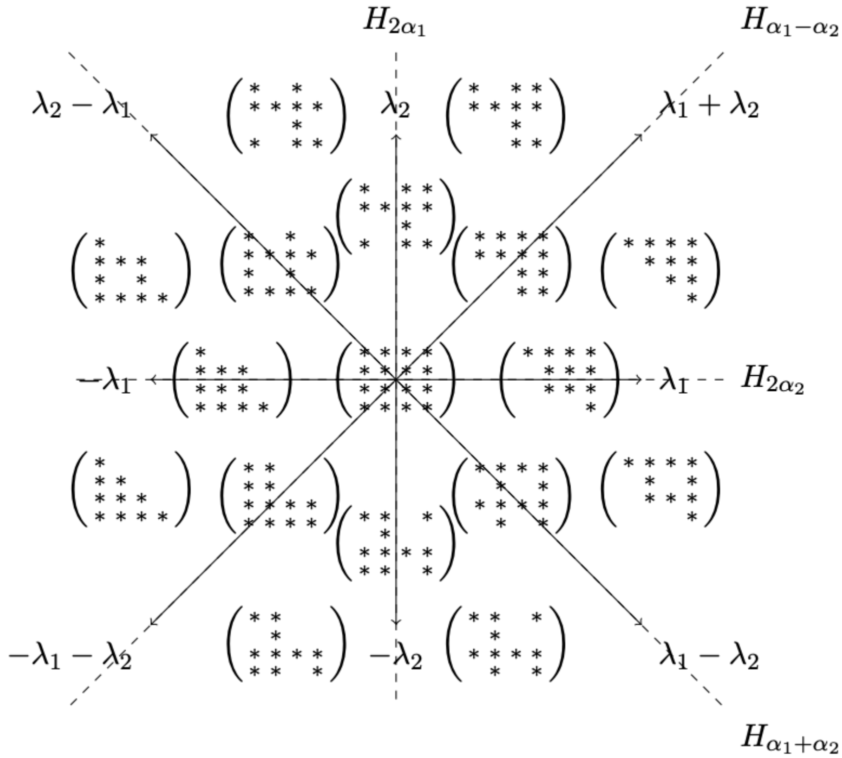
$$\Phi^\vee = \{\pm\lambda_1,\, \pm\lambda_2,\, \pm\lambda_1 \pm \lambda_2\}.$$

Further details may be found in Makisumi.

We will, throughout this section, let ω denote the symplectic form preserved by the action of Sp_4:

$$\omega\left(\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix}\right) = ad' - bc' + cb' - da'$$

8.2. **The Spherical Apartment.** We reproduce Makisumi's illustration:



The Spherical Apartment and Coroots for $\mathbf{Sp}_4$

We see that there are, up to isomorphism, 3 kinds of nontrivial parabolic subgroups:

1) the Borel: $\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$,

2) the Klingen parabolic: $\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$

and

3) the Siegel parabolic: $\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$.

We will center our attention on the Siegel Parabolic. Let

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\},$$

so that

$$U(P) = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Note that in the upper-right 2×2 matrix is actually three-dimensional, and $U(P)$ is isomorphic to the Abelian group $\mathbb{A}^3$:

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

The Levi M, which we shall consider both as a subgroup of Sp_4 and as a quotient of P, is isomorphic to GL_2, and is embedded in Sp_4 via:

$$\left\{ \left(\begin{array}{c|c} m & 0 \\ \hline 0 & m^{-1} \end{array} \right) \right\}.$$

8.3. **Geometry of the Affine Closure.** Now we shall describe the paraspherical space $\overline{G/U(P)}$. We observe there is a map:

$$\mathrm{Sp}_4/U(P) \to \mathrm{Mat}_{4,2}$$

$$[v_1, v_2, v_3, v_4] \left(\begin{array}{c|cc} \mathrm{Id}_2 & \begin{matrix} * & * \\ * & * \end{matrix} \\ \hline 0 & \mathrm{Id}_2 \end{array} \right) \mapsto [v_1, v_2].$$

In the above, the v_i denote column vectors. The image of this map is the set of (v_1, v_2) such that $\omega(v_1, v_2) = 0$ (since they come from the first two columns of an element of Sp_4), and such that v_1 and v_2 are linearly independent. The closure of this image in $\mathrm{Mat}_{4,2}$ drops the linear independence constraint; we let

$$X := \{(v_1, v_2) \in \mathrm{Mat}_{4,2} \, : \, \omega(v_1, v_2) = 0\}$$

denote this closure. Note that X has dimension 7. Since $G/U(P)$ also has dimension $7 = 10 - 3$, we can verify that $G/U(P) \to X$ is an isomorphism onto its image. Moreover, the complement of the this image (i.e., the complement of the rank-2 locus in $X \subset \mathrm{Mat}_{4,2}$) has dimension 5; codimension 2 in X. Thus, Hartog's Lemma implies that X is isomorphic to the affine closure of $G/U(P)$.

Observe that X is a quadric hypersurface in $\mathbb{A}^8$; it is the affine cone over a quadric hypersurface in $\mathbb{P}^7$. Hence it has an isolated conical singularity at the origin.

8.4. **The Slipper Pairing.** We will now consider the Slipper pairing. As we did for $G/U(P)$, we provide a model for $G/U(P^{\mathrm{op}})$ via:

$$\mathrm{Sp}_4/U(P^{\mathrm{op}}) \to \mathrm{Mat}_{4,2}$$

$$[v_1, v_2, v_3, v_4]\left(\begin{array}{cc|c} \multicolumn{2}{c|}{\mathrm{Id}_2} & 0 \\ \hline * & * & \multirow{2}{*}{Id_2} \\ * & * & \end{array}\right) \mapsto [v_3, v_4].$$

Now, observe that $U(P^{\mathrm{op}}) \times U(P)$-invariants of Sp_4 are given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & 0 & 1 \end{pmatrix}\begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}\begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & * & * \\ e & f & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}.$$

Thus the Wang monoid $\overline{M}$ of the Siegel parabolic P is given by $\mathrm{Mat}_2 \cong \operatorname{Spec} k[a, b, e, f]$.

Also, observe that in Sp_4:

$$\begin{pmatrix} * & * & c & d \\ * & * & g & h \\ * & * & k & l \\ * & * & o & p \end{pmatrix}^{-1} = \begin{pmatrix} -p & l & -h & d \\ o & -k & g & -c \\ * & * & * & * \\ * & * & * & * \end{pmatrix}.$$

Thus the Slipper pairing is given by:

$$\mathfrak{S} : G/U(P) \times G/U(P^{\mathrm{op}}) \to \overline{M}$$

$$(gU(P), hU(P^{\mathrm{op}})) \mapsto U^{\mathrm{op}} h^{-1} g U$$

$$\begin{aligned} ((v_1, v_2), (w_1, w_2)) &\mapsto \begin{pmatrix} -\omega(v_1, w_2) & -\omega(v_2, w_2) \\ \omega(v_1, w_1) & \omega(v_1, w_1) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \omega(v_1, w_1) & \omega(v_2, w_1) \\ \omega(v_1, w_2) & \omega(v_2, w_2) \end{pmatrix}. \end{aligned}$$

8.5. **The Function J.** Next we try to compute the function J. The first thing we must do is diagonalize the adjoint representation of $M^\vee$ on $\mathfrak{u}^\vee$. We see that $G^\vee \cong \mathrm{SO}_5$, $M^\vee \cong \mathrm{GL}_2$, and the adjoint representation of M on $\mathfrak{u}$ is $\mathrm{Std} \oplus \det$. The corresponding cocharacters are thus given by λ_1, λ_2 and $\lambda_1 + \lambda_2$. We see that the function associated to the hypergeometric sheaf on T is:

$$\Psi\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \sum_{\substack{s_1 d = t_1 \\ s_2 d = t_2}} \psi(s_1 + s_2 + d)$$
$$= \sum_{d} \psi((t_1 + t_2)/d + d)$$
$$= \mathrm{Kl}(\mathrm{tr}(t))$$

where $\mathrm{Kl}(a) = \sum_{d \in \mathbb{F}_q^\times} \psi(a/d + d)$ is the Kloosterman sum. This function is invariant under W_M, and so descends to a function on T^{rss}/W. So we expect that $J(m) = \mathrm{Kl}(\mathrm{tr}(m))$ for all matrices m in the rss locus of GL_2.

Indeed, we see that pulling back the Kloosterman local system on $\mathbb{A}^1$ to Mat_2 via the trace map provides us with a local system which agrees with the γ-sheaf on the regular semisimple locus of $\mathrm{GL}_2 \subseteq \mathrm{Mat}_2$. Applying the principle of perverse continuation, we find that this sheaf is our γ-sheaf (up to dimension shift and Tate twist); our function (up to a power of q) is $J(m) = \mathrm{Kl}(\mathrm{tr}(m))$.

Now we calculate which functions are special. Indeed, we see that $U(P)^\vee$ has 1-parameter subgroups in SO_5 generated by λ_1, λ_2, and $\lambda_1 + \lambda_2$. We see that $\Pi_P^\vee$ is therefore $\{\lambda_1, \lambda_2\}$, whose sum is $\alpha_P^\vee = \lambda_1 + \lambda_2$. Thus $\alpha_P^\vee(t) = \mathrm{diag}(t, t, t^{-1}, t^{-1})$ which acts on G on the right by scaling the first two columns of g by t. Thus it acts on $\overline{G/U(P)}$ by scaling the corresponding element of $\mathrm{Mat}_{4,2}$, and the space of special functions on $\overline{G/U(P)}$ is given exactly as in (17).

Thus the Fourier transforms between opposite Siegel Parabolics of Sp_4 is our old quadratic Kloosterman Fourier transform fro Section 6. Involutivity then follows from Theorem 6.1.

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